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SPECTRAL CONCENTRATION FOR SELF-ADJOINT OPERATORS

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If the resolvent of a (not necessarily bounded) self-adjoint operator H_κ converges strongly to the resolvent of a self-adjoint operator H, and if λ is an isolated eigenvalue of H of multiplicity $m < \infty$, then although H_κ need not have an eigenvalue near λ, the spectrum of H_κ will in some cases become "concentrated" near λ as κ is reduced. In fact, there exist sets C_κ with Lebesgue measure $o(\kappa^p)$, $p \geqq 0$, such that the spectral projection assigned by H_κ to C_κ converges strongly as $\kappa \to 0$ to the projection on the λ-eigenspace of H, if and only if there exist m pairs $(\lambda_{j\kappa}, \varphi_{j\kappa})$, $j = 1, \cdots, m$, where $\lambda_{j\kappa} \to \lambda$, the $\varphi_{j\kappa}$ are nearly-orthogonal unit vectors converging strongly to the λ-eigenspace, and $||(H_\kappa - \lambda_{j\kappa})\varphi_{j\kappa}|| = o(\kappa^p)$. In this case, C_κ may be taken as the union of intervals about the $\lambda_{j\kappa}$, and the $\lambda_{j\kappa}$ are essentially the only numbers associated in this way with "pseudoeigenvectors" $\varphi_{j\kappa}$ of H_κ. The result is applied to the weak-quantization problem in the theory of the Stark effect, where H is the Hamiltonian operator for the hydrogen atom, and H_κ is the same for the atom in a uniform electric field which vanishes with κ.

In § 1 the basic notions of spectral concentration and pseudoeigenvectors are discussed, and some simple lemmas are proved relating the two. The theorem quoted above is proved in § 2 (Theorem 2.7), and the question arises how the pairs $(\lambda_{j\kappa}, \varphi_{j\kappa})$ can be constructed. For a family of the form $H_\kappa = H + \kappa V$, this construction is carried out in § 3 by means of the formal perturbation process applied to the unperturbed eigenvalue λ. In § 4, the special case is considered in which λ is *stable*, i.e. H_κ has m eigenvalues in a neighborhood of λ; asymptotic estimates of these perturbed eigenvalues follow easily. Finally, in § 5, the theory is applied to the generic example, the family of operators appearing in the Stark effect. Here the spectrum of H_κ is purely continuous and covers the real line, and so technically the perturbed system has no stationary states. Yet the lines in the physical spectrum of hydrogen persist when a weak electric field is applied, each one splitting into several quite sharp lines. These lines can be traced to the existence of "almost stationary" states, which are represented by the pseudoeigenvectors $\varphi_{j\kappa}$ mentioned above. It is stressed that there are many other vectors ψ_κ which represent "almost stationary" states, in the sense that the solution of the equation of motion with initial state ψ_κ remains close to ψ_κ for a long time. It is the fact that no sequence $\varphi_{j\kappa_n}$ converges weakly to zero $(\kappa_n \to 0)$