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OPERATOR VALUED ANALYTIC FUNCTIONS AND GENERALIZATIONS OF SPECTRAL THEORY

LOTHROP MITTENTHAL

This paper is concerned with an analytic operator valued function $F(\lambda)$ acting upon a Banach space X, where $F(\lambda)$ is bounded and $F(\lambda)F(\mu) = F(\mu)F(\lambda)$ for all $\lambda, \mu \in \Delta$ where Δ is the domain of analyticity of $F(\lambda)$. The singular set of $F(\lambda)$ is analogous to the spectrum of a single operator. In the case of the single operator, employing the corresponding resolvent operator, a number of interesting properties are known to be associated with the spectral sets. These include projections and homomorphisms between scalar valued analytic functions and functions of the operator. This paper considers a suitable generalization of the resolvent operator and which properties of spectral sets carry over to open and closed subsets of the singular set of the operator valued analytic function.

It is shown that a suitable generalization of the resolvent operator is $F'(\lambda)F(\lambda)^{-1} = F(\lambda)^{-1}F'(\lambda)$, from our assumed commutativity, where $F'(\lambda) = (d/d\lambda)F(\lambda)$. In addition, it is shown that certain proper open and closed subsets of the singular set, termed separating subsets, have many of the properties of spectral sets. These properties include a relation between ascent and descent of the operator and the order of the pole of the generalized resolvent, projections analogous to spectral projections, and an operational calculus. A sufficient condition for a singular subset to be separating is derived. In addition, a new operator is defined which in a sense represents the $F(\lambda)$ on the subspace corresponding to a given separating singular subset.

DEFINITION. The singular set of $F(\lambda)$, $S(F(\lambda))$ or $S(F)$, is the set of all $\lambda \in \mathscr{C}$, the complex plane, such that $F(\lambda)$ is not continuously invertible. The complement of $S(F)$ will be called the regular set, $R(F(\lambda))$ or $R(F)$.

In this paper, it will be assumed that $S(F)$ is bounded and that $S(F) \subset \Delta$. In particular, this will include polynomials with operator coefficients, e.g. $F(\lambda) = \lambda^n I + \lambda^{n-1}A_1 + \cdots + A_n$ where the A_j are all bounded commuting operators. It has been shown by A. Taylor [3, p. 590] that on $R(F) \cap \Delta$, $F(\lambda)^{-1}$ is also analytic. As in the case of the spectrum of a single operator, it is easy to show that $R(F)$ is open and that $S(F)$ is closed.

If, for example, we choose $F(\lambda) = \lambda^n I + \lambda^{n-1}A_1 + \cdots + A_n$ and