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FUNCTIONS WITH REAL POLES AND ZEROS

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Throughout this paper $\{\lambda_n\}$ is a real sequence with $\lambda_n \neq 0$ and $\lambda_n \leqq \lambda_{n+1}$, $-\infty < n < \infty$. The counting function $\Lambda_1(u)$ is the number of λ_n between 0 and u, counted negatively for negative u. Similarly μ_n is a real sequence with $\mu_n \neq 0$ and with counting function $\Lambda_2(u)$. In this summary (which corresponds to the case $p = 1$ of the paper) we define

$$F(z) = \prod \frac{(1 - z/\lambda_n)e^{z/\lambda_n}}{(1 - z/\mu_n)e^{z/\mu_n}}$$

by taking all factors for which λ_n and μ_n lie on the interval $(-R, R)$ and then letting $R \to \infty$. Our objective is to obtain conditions on the growth of $F(z)$ from conditions on the function $\Lambda(u) = \Lambda_1(u) - \Lambda_2(u)$.

Denoting the even part of $\Lambda(u)$ by $\Lambda_e(u)$, we can state our first result as follows: Suppose $\Lambda(u) = O(u)$ and

$$\lim_{r\to\infty} \left(\frac{\Lambda(ur)}{ur} - \frac{\Lambda(r/u)}{r/u} \right) = 0$$

for each $u \neq 0$. Then

$$\log |F(re^{i\theta})| = \pi\Lambda(r) |\sin\theta| - 2\Lambda_e(r)\theta \sin\theta + r\cos\theta \int_{-r}^{r} \frac{\Lambda(u)}{u^2} du$$

apart from an error term $\eta r \log |2 \csc\theta|$, where $\eta \to 0$ uniformly in $0 < |\theta| < \pi$ as $r \to \infty$. This improves theorems of Pfluger, Kahane and Rubel, Cartwright, and others, in that we do not assume existence of $\lim \Lambda(u)/u$, we do not assume that F is entire or even, and the error term has a convergent integral with respect to θ. Similar theorems for functions with negative poles and zeros, given later in the paper, generalize other familiar results. Here the error term involves $\log (2 \sec \frac{1}{2}\theta)$.

Another kind of result is briefly described as follows: Let $R = R(x)$ and $S = S(x)$ be positive functions such that the ratios x/R, R/x, x/S, S/x are bounded as $|x| \to \infty$. Then for many purposes the function

$$\log F(z) - z \int_{-|x|}^{|x|} \frac{\Lambda(u)}{u^2} du$$

can be replaced by the function

$$\Lambda^*(z; R, S) = \int_{x-R}^{x+S} \frac{\Lambda(u)}{z-u} du \ .$$

This remark is given content by detailed estimates of the error and of Λ^*. (For simplicity of statement the text takes $R = S$ but the form mentioned here is sometimes more con-