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INTEGRALS WHICH ARE CONVEX FUNCTIONALS

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This paper examines numerical functionals defined on function spaces by means of integrals having certain convexity properties. The functionals are themselves convex, so they can be analysed in the light of the theory of conjugate convex functions, which has recently undergone extensive development. The results obtained are applicable to Orlicz space theory and in the study of various extremum problems in control theory and the calculus of variations.

In everything that follows, let T denote a measure space with a σ-finite measure dt. Let L be a particular real vector space of measurable functions u from T to R^n (for a fixed n). For instance, one could take L to be the space $L_n^p(T)$ consisting of all R^n-valued measurable functions u on T such that $\Phi_p(u) < +\infty$, where

$$\Phi_p(u) = \int_T \varphi_p(u(t))dt \quad \text{and} \quad \varphi_p(x) = (1/p)\,|\,x\,|^p, \ 1 \leqq p < +\infty$$

with $|\cdot|$ denoting the Euclidean norm on R^n. No matter which L is chosen, one can regard Φ_p as a functional from L to $(-\infty, +\infty]$. Then Φ_p is convex, in consequence of the fact that the function φ_p is convex on R^n. (A function F from a real vector space to $(-\infty, +\infty]$ is said to be *convex* if

$$F(\lambda x + (1-\lambda)y) \leqq \lambda F(x) + (1-\lambda)F(y)$$

always holds when $0 < \lambda < 1$.) Notice that, if φ_∞ is the convex function defined by

$$\varphi_\infty(x) = \lim_{p\to\infty} \varphi_p(x) = \begin{cases} 0 & \text{if} \quad |\,x\,| \leqq 1\,, \\ +\infty & \text{if} \quad |\,x\,| > 1\,, \end{cases}$$

the corresponding integral $\Phi_\infty(u)$ is finite if and only if u belongs to the unit ball of the space $L_n^\infty(T)$ of essentially bounded measurable functions.

Here we propose to study a much broader class functionals than the Φ_p, $1 \leqq p \leqq \infty$. These functionals are of the form

$$I_f(u) = \int_T f(t, u(t))dt \quad \text{for} \quad u \in L\,,$$

where f is a function from $T \times R^n$ to $(-\infty, +\infty]$, such that $f(t, x)$ is a convex function of $x \in R^n$ for each $t \in T$. Such a function f we call a *convex integrand* for convenience.

As a preliminary task, we must come up with conditions on f