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CHARACTERISTIC POLYNOMIALS OF SYMMETRIC MATRICES

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Let F be a field and p an F-polynomial. We say that p is F-real if and only if every real closure of F contains the splitting field of p over F. Our main purpose is to prove

THEOREM 1. **Let F be an algebraic number field and p a monic F-polynomial with an odd degree factor over F. Then p is F-real if and only if it is the characteristic polynomial of a symmetric F-matrix.**

That p must be F-real follows from work of Krakowski [4, Satz 3.3]. To prove the coverse we generalize results of Sapiro [6] in Lemma 1 and Theorem 3. Sapiro deals with the case in which p is a cubic. Theorem 4 considers the minimum dimension of symmetric matrices with a given root.

2. A basic lemma.

In our proof we shall study congruence classes of certain symmetric matrices which are defined below. We shall denote congruence of the matrices A and B over the field F (i.e., $A = TBT'$ for some nonsingular F-matrix T) by $A \sim B(F)$.

DEFINITION. Let G be a field with subfield F. If $\lambda \in G$ is nonzero and if $\alpha_1, \cdots, \alpha_n$ form a basis for G (as a vector space) over F, define the matrices $M = \| \alpha_i^{(j)} \|$ and $D(\lambda) = \operatorname{diag}(\lambda^{(1)}, \cdots, \lambda^{(n)})$ where superscripts denote conjugacy over F. We call

$$A = A(\lambda) = MD(\lambda)M'$$

a *matrix from G to F*. Clearly

$$a_{ij} = \operatorname{tr}_{G/F}(\lambda \alpha_i \alpha_j) .$$

If $\mathscr{A} = \Sigma \oplus G_i$ where the G_i are extension fields of F, and if A_i is a matrix from G_i to F, then any matrix congruent to $\Sigma \oplus A_i$ over F is called a matrix from $\mathscr{A}$ to F. Note that a different choice for the basis $\alpha_1, \cdots, \alpha_n$ would lead to a matrix congruent to $A(\lambda)$ over F.

LEMMA 1. *Let F be a field and $p = q_1 \cdots q_m$ a monic F-polynomial decomposed into prime factors over F. Assume that the splitting field of p over F is a separable extension of F. If the identity is a matrix from*

$$\sum_1^m \oplus F[x]/(q_i))$$