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DUAL GROUPS OF VECTOR SPACES

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Let E be a topological vector space over a field K having a nontrivial absolute value. Let E' be the dual space of continuous linear maps $E \to K$, and $\hat{E}$ the dual group of continuous characters $E \to R/Z$. $\hat{E}$ is a vector space over K by $(a\varphi)(x) = \varphi(ax)$, and composition with a nonzero character of K is a linear map of E' into $\hat{E}$. This map is always an isomorphism if K is locally compact, while if K is not locally compact it is never an isomorphism unless $\hat{E} = 0$. When K is locally compact, E' is in addition topologically isomorphic to $\hat{E}$ if each is given its topology of uniform convergence on compact sets. This leads to conditions on E which imply that E is topologically isomorphic to $(\hat{E})\hat{}$.

THEOREM 1. *Let K be a field with absolute value. Then $\hat{K}$ is one-dimensional over K if and only if K is locally compact.*

Proof. The sufficiency of local compactness is fairly well known (cf. [4, p. 92–3] for the characteristic zero case). To prove it, one takes a nonzero character π of K and considers the subspace $K\pi$ of $\hat{K}$. It is easy to check that $a \mapsto a\pi$ is a bicontinuous linear map, so $K\pi$ is complete and hence closed in $\hat{K}$. On the other hand, $K\pi$ separates the points of K, so by Pontrjagin duality it is dense in $\hat{K}$. Thus $\hat{K} = K\pi$.

Suppose conversely that $\hat{K}$ is one-dimensional, and choose a nonzero π in $\hat{K}$. The completion of K will again be a field, say L, and π extends to a character of L. Then every $a \in L$ gives a character $a\pi$ of L. If $a \neq b$, then $a - b$ is invertible, and so $\pi((a-b)c)$ cannot be zero for all c. Thus no two of the characters $a\pi$ are equal, and hence no two can agree on the dense set K. This contradicts one-dimensionality of $\hat{K}$ unless $K = L$, and we conclude that K must be complete. Hence if K is archimedean, it is locally compact.

We now assume that K is nonarchimedean. Let $A = \{x: |x| \leqq 1\}$, $M = \{x: |x| < 1\}$. Let π be a character of the discrete group A/M with $\pi(1) \neq 0$; we extend π to a character of the discrete group K/M and interpret it as an element of $\hat{K}$. Let $c > 1$ be an element of the value group, and consider the group G_c/M, where $G_c = \{x: |x| \leqq c\}$. All characters of this discrete group extend to characters of K vanishing on M, and by one-dimensionality they all come from multiples of π.

Now if $a \in A$, then $aM \subset M$, so $a\pi$ vanishes on M; conversely, if $a\pi$ vanishes on M, then $1/a \notin M$ and $a \in A$. Similarly, $a\pi$ vanishes