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IDEAL EXTENSIONS OF SEMIGROUPS

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An ideal extension (here called an extension) of a semigroup S by a semigroup with zero Q is a semigroup V such that S is an ideal of V and the Rees quotient semigroup V/S is isomorphic to Q. To study the structure of these extensions, special kinds of extensions are introduced, called strict and pure extensions. It is proved that any extension of S is a pure extension of a strict extension of S; also, if Q has no proper nonzero ideals, any extension of S by Q is either strict or pure. Dense extensions, closely related to Ljapin's "densely embedded ideals", are special cases of pure extensions. When S is weakly reductive, constructions of strict, pure, and arbitrary extensions of S are given, including descriptions of the ramification function.

Extensions were first systematically studied by Clifford [1] who gave the first general structure theorem in the case when S is weakly reductive (Theorem 4.21 of [2]) (later extended to arbitrary S by Yoshida [7]). In this theorem the multiplication in the extension V of S by Q is described in terms of the action of V on S and a ramification function. Our structure theorems are a refinement of this in that the ramification function is not used explicitly, or, equivalently, is described in terms of other functions. Our methods are not essentially new; except in § 3, we use exclusively the action of the extension V on S, this gives rise also to the notions of strict and pure extensions: the extension V of S is strict if every element of $V-S$ has same action on S as some element of S, pure if no element of $V-S$ has this property.

In the introductory § 1 we establish some preliminary results concerning extensions of an arbitrary semigroup S using the translational hull of S and introduce the notion of the type of an extension. This material is used in § 2 where we introduce strict and pure extensions, study their main properties and construct them in the case when S is weakly reductive. In § 3 we interpret some results of § 2 by means of congruences on an extension V of S whose restriction to S is the equality relation on S. This is particularly suitable for the study of dense extensions and leads in particular to another proof of a theorem of Gluskin concerning dense embeddings. Section 4 contains our main results; we construct all extensions of a weakly reductive semigroup S by an arbitrary semigroup with zero Q and establish when two such extensions are equivalent in the sense of Clifford.