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OSCILLATION CRITERIA FOR ELLIPTIC EQUATIONS

V. B. HEADLEY AND C. A. SWANSON

Conditions on the coefficients of a linear elliptic partial differential equation will be obtained which are sufficient for the equation to be oscillatory in certain unbounded domains. The criteria obtained in the first three theorems involve integrals of suitable majorants of the coefficients while the criterion in Theorem 4 involves limits of these majorants at infinity. We also obtain a nonoscillation criterion involving similar limits.

Oscillation criteria of both limit type and integral type will be obtained for the linear elliptic partial differential equation

$$(1) \qquad Lu \equiv \sum_{i,j=1}^{n} D_i(a_{ij}D_ju) + bu = 0$$

in unbounded domains R in n-dimensional Euclidean space E^n. Our theorems constitute extensions of several well-known one-dimensional oscillation theorems of Kneser-Hille [6] (limit type), Leighton [8], Moore [10], and Wintner [13] (integral type). A special case of Theorem 4 below was obtained by Glazman [4, 5] when L is the Schrödinger operator and R coincides with E^n. Analogues of Theorem 1 were obtained by Kreith [7] and Swanson [12] in the case that one variable is separable and R is limit cylindrical, i.e., contains an infinitely long cylinder.

Points in E^n are denoted by $x = (x^1, x^2, \cdots, x^n)$ and differentiation with respect to x^i is denoted by D_i, $i = 1, 2, \cdots, n$. The functions a_{ij} and b involved in (1) are assumed to be real-valued and continuous on $R \cup \partial R$, and the matrix (a_{ij}) is supposed to be symmetric and positive definite in R (ellipticity condition). A "solution" of (1) is defined in the usual way [1, 12].

We assume that R contains the origin and that R is large enough at ∞ in the x^n direction to contain the cone $C_\alpha = \{x \in E^n : x^n \geqq |x| \cos \alpha\}$ for some α, $0 < \alpha \leqq \pi$. The boundary ∂R of R is supposed to have a piecewise continuous unit normal vector at each point. The following notations will be used:

$$R_r = R \cap \{x \in E^n : |x| > r\} ; \qquad S_r = \{x \in R \cup \partial R : |x| = r\} .$$

A bounded domain $N \subset R$ is said to be a *nodal domain* of a nontrivial solution u of (1) if and only if $u = 0$ on ∂N. The differential equation (1) is said to be *oscillatory* in R if and only if there exists a nontrivial solution u_r of (1) with a nodal domain in R_r for