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# THE $\delta^2$-PROCESS AND RELATED TOPICS II

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**This paper considers three transforms of a complex series $\Sigma a_n$: namely, (1) Aitken's $\delta^2$-transform $\Sigma b_n$, (2) Lubkin's $W$-transform $\Sigma c_n$, and (3) a closely related transform $\Sigma d_n$ which the author calls the $W1$-transform and for which $\sum_0^n d_k = \sum_0^{n+1} c_k$. If $a_{n-1} \neq 0$, set $r_n = a_n/a_{n-1}$. If, moreover, $\Sigma a_n$ converges, define $T_n = (a_n + a_{n+1} + \cdots)/a_{n-1}$ and let $MR(\Sigma a_n)$ be the class of all series converging more rapidly to the sum $S = \Sigma a_n$ than $\Sigma a_n$. Some of the results proven in this paper are as follows:**

**(1) If $b_n/a_n \to 0$, then the three conditions (i) $\Sigma b_n \in MR(\Sigma a_n)$, (ii) $\Sigma c_n \in MR(\Sigma a_n)$, and (iii) $\Sigma d_n \in MR(\Sigma a_n)$ are equivalent.**

**(2) $\Sigma b_n \in MR(\Sigma a_n)$ if and only if $\Delta T_n \to 0$.**

**(3) If $|r_n| \leqq \rho < 1$ for all sufficiently large $n$, then the three conditions (i) $\Sigma b_n \in MR(\Sigma a_n)$, (ii) $\Delta r_n \to 0$, and (iii) $b_n/a_n \to 0$ are equivalent.**

Samuel Lubkin has given several sufficient conditions for $\Sigma b_n \in MR(\Sigma a_n)$ in case $\Sigma a_n$ is a real series. The third result above contains a generalization of one of his results to the complex plane while relaxing some of his hypothesis.

The following results on complex products are also proven:

(4) If the sequence $\{1/a_n - 1/a_{n-1}\}$ is bounded, then the product $\Pi_0^\infty (1 + a_n)$ diverges.

(5) Suppose that $|r_n| \leqq \rho < 1$ for all sufficiently large $n$ and $a_n \neq -1$ for all $n$. Then a necessary and sufficient condition for the $\delta^2$-transform to accelerate the convergence of the infinite product $\Pi_0^\infty (1 + a_n)$ is that $\Delta r_n \to 0$.

The notations and definitions set forth in Tucker [2] will be used in this paper. In particular, $S_n = a_0 + a_1 + \cdots + a_n$, $\Sigma a_n = \sum_0^\infty a_n$, and $S = \Sigma a_n$ if $\Sigma a_n$ is convergent. Given a second series $\Sigma a'_n$ we use the notation $S'_n = a'_0 + \cdots + a'_n$, $r'_n = a'_n/a'_{n-1}$ for $a'_{n-1} \neq 0$, $S' = \Sigma a'_n$ and $T'_n = (S' - S'_{n-1})/a'_{n-1}$ for $a'_{n-1} \neq 0$. Likewise, given a "transform sequence" $\{\alpha_n\}$, $\alpha_n$ complex, we set $S_{\alpha n} = S_n + a_{n+1}\alpha_{n+1}$ for $n \geqq 0$, $a_{\alpha 0} = S_{\alpha 0} = a_0 + a_1\alpha_1$, and $a_{\alpha n} = S_{\alpha n} - S_{\alpha(n-1)}$ for $n \geqq 1$.

The transform sequences associated with the $\delta^2$, $W$, and $W1$ transforms are defined respectively as follows:

(i) $\alpha_n = 1/(1 - r_n)$, $n \geqq 1$,

(ii) $\alpha_1 = -a_0/a_1$; $\alpha_n = (1 - r_{n-1})/(1 - 2r_n + r_{n-1}r_n)$, $n \geqq 2$,

(iii) $\alpha_n = (1 - r_{n+1})/(1 - 2r_{n+1} + r_n r_{n+1})$, $n \geqq 1$.

Whenever division by zero occurs in (i), we set $\alpha_n = 0$. We do likewise for (ii) and (iii). As in Tucker [2], we retain the notation