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REPRESENTABLE DISTRIBUTIVE NOETHER LATTICES

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Recently, Bogart showed that a certain class of distributive Noether lattices, namely regular local ones, are embeddable in the lattice of ideals of an appropriate Noetherian ring. In this paper a characterization of the distributive Noether lattices which are representable as the complete lattice of ideals of a Noetherian ring is obtained.

We observe that if $L(R)$ is the lattice of ideals of a ring R (commutative with 1) and if A, B and C are elements of $L(R)$ with $A \nleq B$ and $A \nleq C$, then there exists a principal element $E \in L(R)$ with $E \leqq A$, $E \nleq B$ and $E \nleq C$. If a Noether lattice L has this property, then we will say that L satisfies the *weak union condition*. (The term union condition has been used elsewhere for a stronger property.) With this definition, then, the main result of this paper is that a distributive Noether lattice L is representable as the lattice of ideals of a Noetherian ring if, and only if, L satisfies the weak union condition.

We adopt the terminology of [2] and we assume throughout that L is a Noether lattice.

LEMMA 0. *If L is local, and if the maximal element $P \in L$ is principal, then every element $A \neq 0$ of L is a power $P^n (0 \leqq n)$ of P.*

Proof. If $A \neq 0$, then by the Intersection Theorem [2] there exists a largest integer n such that $A \leqq P^n$. Then

$$A = A \wedge P^n = (A\colon P^n)P^n ,$$

so since $A \nleq P^{n+1}$, it follows that $A\colon P^n = I$, and therefore that $A = P^n$.

LEMMA 1. *Assume L is distributive and satisfies the weak union condition. If L is local and if the maximal element of L is principal, or if 0 is prime and every element $A \neq 0$ has a primary decomposition involving only powers of maximal primes, then L is representable as the lattice of ideals of a Noetherian ring.*

Proof. Assume L is local with maximal element P, and that P is principal. Let (R, M) be a regular local ring of altitude one. If 0 is prime in L, then the powers of P are distinct, and L is isomorphic to the lattice of ideals of R. If 0 is not prime in L, and if k is the least positive integer such that $P^k = P^{k+1}$, then L is isomorphic to the lattice of ideals of $R \mid M^k$.