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AN APPLICATION OF A NEWTON-LIKE METHOD TO THE EULER-LAGRANGE EQUATION

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It is known that any function which minimizes a functional of the form $J(y) = \int_a^b f(x, y, y')$ and satisfies prescribed boundary values must be a solution of the corresponding Euler-Lagrange equation: $f_3(x, y, y') - \int_a^x f_2(x, y, y') = c$. Let us call any equation of the form: $g(x, y, y') - \int_a^x h(x, y, y') = c$ a generalized Euler-Lagrange equation.

In this paper we propose a Newton-like method and show that this proposed method is general enough to enable us to construct solutions of the generalized Euler-Lagrange equation.

Let X and Y be Banach spaces, Ω an open subset of X and $P: \Omega \to Y$. By $[X, Y]$ we mean the Banach space of all bounded linear operators with the usual operator norm, by P' the first derivative of P and by P'' the second derivative of P. The class of all functions defined on Ω which have continuous derivatives up to and including order n at each point of Ω is denoted by $C^n(\Omega)$. Distinct elements of $C^n(\Omega)$ may have totally distinct ranges depending on the application. The distinction between Gateaux and Fréchet is unnecessary since the derivatives will be continuous.

2. **The weak Newton sequence.** Let H and Y be Banach spaces, Ω a nonempty open subset of H and $P: \Omega \to Y$.

DEFINITION. Given $x_0 \in \Omega$ the sequence $\{x_n\}_0^\infty$,

$$x_{n+1} = x_n - [P'(x_n)]^{-1}P(x_n) ,$$

is called the Newton sequence for x_0 (with respect to P).

DEFINITION. Given $x_0, \bar{x} \in \Omega$ the sequence $\{x_n\}_0^\infty$,

$$x_{n+1} = x_n - [P'(\bar{x})]^{-1}P(x_n) ,$$

is called the modified Newton sequence for x_0 at $\bar{x}$ (with respect to P). When $x_0 = \bar{x}$ we say simply the modified Newton sequence for x_0.

REMARK. The Newton sequences exist if and only if $P'(\bar{x})$ and $P'(x_n)$ exist and are invertible and $x_n \in \Omega$ for all n.