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SOME ISOPERIMETRIC INEQUALITIES FOR THE EIGENVALUES OF VIBRATING STRINGS

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If a string with integrable density function $p(x)$ is fixed at the points $x = 0$, $x = a$ then the natural frequencies of vibration are determined by the eigenvalues of the Sturm-Liouville System

$$(1) \qquad y'' + \lambda p(x)y = 0 \qquad y(0) = y(a) = 0 .$$

These eigenvalues depend on the density function $p(x)$ and we denote them accordingly by $\lambda_n(p)$,

$$0 < \lambda_1(p) < \lambda_2(p) < \cdots .$$

In this work we investigate the nature of the density functions which yield the largest and smallest possible value for $\lambda_n(p)$ assuming that the average value of the density $p(x)$ defined by

$$P(x) = \frac{1}{x}\int_0^x p(\zeta)d\zeta$$

is restricted in some manner.

We assume for example that $P(x)$ is decreasing or that $P(x)$ is concave (see Theorems 4 and 7 below).

Assuming a string of given mass m and a bounded density function $p(x)$, $0 \leqq p(x) \leqq H$, M. G. Krein [8] has obtained the sharp bounds

$$\frac{4Hn^2}{m^2} X\left(\frac{m}{aH}\right) \leqq \lambda_n(p) \leqq \frac{\pi^2 n^2 H}{m^2} ,$$

where $X(t)$ is the smallest positive root of the equation

$$\sqrt{X} \tan \sqrt{X} = \frac{t}{1-t} .$$

Banks [1], [2], [5] has obtained some improvements of the Krein inequality by imposing various restrictions on the density function $p(x)$. Schwarz [12], Nehari [10], [11], Banks [4] and Maki [9] have obtained additional related results.

Given numbers m, H, a such that $m < aH$, and an integrable density functions $p(x)$ defined on $[0, a]$ for which

$$(2) \qquad 0 \leqq p(x) \leqq H , \qquad \int_0^a p(x)dx = m ,$$