PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 1, 1969

COUNTABLE RETRACING FUNCTIONS AND Π_2^0 PREDICATES

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In this paper our attention centers on partial recursive retracing functions, especially *countable* **ones** (*as defined below*), **and on their relationship with classes of number theoretic functions constituting solution sets for Π_2^0 function predicates in the Kleene hierarchy. Arithmetical function predicates which have** *singleton* **solution sets (i.e., so called** *implicit arithmetical definitions*) **have received ample attention in the recursion-theoretic literature. We shall be concerned with such predicates, at the levels Π_1^0 and Π_2^0; but we shall primarily be concerned with the wider classes of Π_1^0 and Π_2^0 predicates having** *countable* **solution sets. In § 5, we show (by obtaining examples which range over the whole of $\mathscr{H} \cap \{D \mid D > 0'\}$, $\mathscr{H}$ as defined in § 4) that a solution of a countable Π_1^0 predicate need not be definable by means of a "strong" Π_2^0 predicate; in fact, we establish the corresponding (slightly stronger) proposition for countable,** *finite-to-one*, **general recursive retracing functions. The question whether all solutions of a countable Π_2^0 predicate are Π_2^0 definable is left open but subjected to conjecture.**

In § 4, we present a new and somewhat more compact proof for one of the main theorems obtained by C. E. M. Yates in [20] (indeed, we obtain a slightly stronger theorem); and we shall derive one of the other principal results of [20] as a corollary to some of our theorems. In § 4 and § 5 systematic use is made of the main content of Myhill's paper [14].

We proceed now to lay down the conventions which are to be in force throughout the rest of the paper; at the end of this section we shall indicate briefly the contents of each of the remaining sections. The symbol N always denotes the set $\{0, 1, 2, \cdots\}$ of natural numbers. We shall in general use lower case Greek letters for subsets of N and lower case Latin letters for functions (partial or total) with domain and range included in N, although this particular convention will not be adhered to with absolute rigor. Given a function $f: \alpha \rightarrow N$ where $\alpha \subseteq N$, we denote by δf the domain, α, of f, and by ρf the range of f. We fix a standard recursive enumeration ([10]) of the partial recursive functions of one variable, and denote this enumeration by $\{\varphi_e\}_{e=0}^{\infty}$; similarly, we fix a standard recursive enumeration $\{\varphi_e^2\}_{e=0}^{\infty}$ of the partial recursive functions of two variables. We further fix a recursive enumeration $\mathscr{E}_1$ of the set $\{(e, x, y) \mid \varphi_e(x) = y\}$; and we denote