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TWISTED COHOMOLOGY AND ENUMERATION OF VECTOR BUNDLES

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In the present paper we give a technique for completely enumerating real 4-plane bundles over a 4-dimensional space, real 5-plane bundles over a 5-dimensional space, and real 6-plane bundles over a 6-dimensional space. We give a complete table of real and complex vector bundles over real projective space P_k, for $k \leqq 5$. Some interesting results are:

(0.1.1.) Over P_5, there are four oriented 4-plane bundles which could be the normal bundle to an immersion of P^5 in R^9, i.e., have stable class $2h + 2$, where h is the canonical line bundle. Of these, two have a unique complex structure.

(0.1.2.) Over P_5 there is an oriented 4-plane bundle which we call C, which has stable class $6h - 2$, which has two distinct complex structures. D, the conjugate of C, i.e., reversed orientation, has no complex structure.

(0.1.3) Over P_5, there are no 4-plane bundles of stable class $5h - 1$ or $7h - 3$.

0.2. In reading the tables (4.5.2) and (4.6), remember that if ξ: $P_k \rightarrow BO(n)$ or $\xi: P_k \rightarrow BU(n)$ is a locally oriented (i.e., oriented over base-point) real or complex vector bundle, and if

$$a \in H^k(P_k; \pi_k(BO(n), \xi))$$

(local coefficients if ξ unoriented) or $a \in H^k(P_k; \pi_k(BU(n))$, then $\xi + a$ is a vector bundle obtained by cutting out a disk in the top cell of P_k and joining a sphere with some vector bundle on it.

0.3. Since some of the homotopy groups of $BO(n)$ are acted upon nontrivially by $Z_2 \cong \pi_1(BO(n))$ for n even, we study cohomology with local coefficients in § 3.

1.2. From here on, we assume that all spaces are connected C. W.-complexes with base-point, all maps are b.p.p. (base-point-preserving) and that all homotopies are b.p.p.

For any space Y, we choose a Postnikov system for Y, that is: for each integer $n \geqq 0$, a space $(Y)_n$ and a map $P_n: Y \rightarrow (Y)_n$ which induces an isomorphism in homotopy through dimension n, where all homotopy groups of $(Y)_n$ are zero above n; for each $n \geqq 1$ a fibration $p_n: (Y)_n \rightarrow (Y)_{n-1}$ such that $p_n P_n = P_{n-1}$. The fiber of each p_n is then an Eilenberg-MacLane space of type $(\pi_n(Y), n)$. If X is a space of finite dimension m, then $[X; Y]$, the set of homotopy classes of maps