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EXTENSIONS OF A FOURIER MULTIPLIER THEOREM OF PALEY

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Let A be the class of continuous power series on the unit circle T, that is those continuous functions f whose Fourier coefficients $\hat{f}(n)$ are 0 for negative indices n. It is known that the most that can be said about the size of the coefficients of such f is that they are square summable. For instance Paley proved the following: Suppose that $\sum_0^\infty |w(n)|^2 = \infty$. Then there is an f in A with $\sum_0^\infty |\hat{f}(n)w(n)| = \infty$. In other words the l^2 sequences are the only multipliers which map A into the class of absolutely convergent power series.

The main result of this paper is that Paley's theorem can be generalized as follows: Let G be a compact Abelian group with a partially ordered dual group Γ. Denote by A the class of continuous functions f on G whose Fourier coefficients $\hat{f}(\gamma)$ vanish off the non-negative cone S of Γ. Let E be a totally ordered subset of S and w be a function defined on E which is not square summable. Then $\sum_E |\hat{f}(\gamma)w(\gamma)| = \infty$ for some f in A.

The class A when Γ is in fact a totally ordered group is a frequently considered generalization of the algebra of continuous power series. In this situation S itself is totally ordered so that $\sum_S |w(\gamma)|^2 < \infty$, whenever $\sum |\hat{f}(\gamma)w(\gamma)| < \infty$ for all f in A. This was obtained for $G = T^n$ by Helson [4] and in general by Rudin [8, p. 222]. Their proofs differed from Paley's although his method can be made to work in the situations they considered.

Now the power series discussed in the first paragraph are the restrictions to the circle of those functions which are continuous on the closure of the unit disc and analytic in its interior. From this point of view it would be natural, when $G = T^2$, to let A be the class of restrictions, to the distinguished boundary of the unit bidisc, of functions which are continuous on the closure and analytic in the interior of the bidisc. These are precisely the continuous functions on T^2 whose Fourier coefficients $\hat{f}(m, n)$ vanish off the first quadrant S of Z^2. The full analogue of Paley's theorem would be that every sequence w with the Paley multiplier property, $\sum |w(N)\hat{f}(N)| < \infty$ for all f in A, is square summable.

It is not known whether this strong version of theorem holds. The Helson-Rudin proofs for the case when S is a half space depend on a property of the analytic projection L taking trigonometric polynomials $\sum_\Gamma \hat{f}(\gamma)\gamma(x)$ into $\sum_S \hat{f}(\gamma)\gamma(x)$. Specifically, $\|Lf\|_p \leqq K_p \|f\|_1$