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LINEAR TRANSFORMATIONS OF TENSOR PRODUCTS PRESERVING A FIXED RANK

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In this paper T is a linear transformation from a tensor product $X \otimes Y$ into $U \otimes V$, where X, Y, U, V are vector spaces over an infinite field F. The main result gives a characterization of surjective transformations T for which there is a positive integer k ($k < \dim U$, $k < \dim V$) such that whenever $z \in X \otimes Y$ has rank k then also $Tz \in U \otimes V$ has rank k. It is shown that $T = A \otimes B$ or $T = S \circ (C \otimes D)$ where A, B, C, D are appropriate linear isomorphisms and S is the canonical isomorphism of $V \otimes U$ onto $U \otimes V$.

Let F be an infinite field and X, Y, U, V vector spaces over F. We denote by T a linear transformation of the tensor product $X \otimes Y$ into $U \otimes V$. The rank of a tensor $z \in X \otimes Y$ is denoted by $\rho(z)$. By definition $\rho(o) = 0$. The subspace of X spaned by the vectors $x_1, \cdots, x_n \in X$ will be denoted by $< x_1, \cdots, x_n >$.

Lemma 1. *Let k be a positive integer such that $z \in X \otimes Y$ and $\rho(z) = k$ imply that $\rho(Tz) = k$. Then $\rho(z) \leqq k$ implies that $\rho(Tz) \leqq k$ for all z.*

Proof. If this is not true then for some $z \in X \otimes Y, z \neq 0$, we have $\rho(z) < k$ and $\rho(Tz) > k$. There exists $t \in X \otimes Y$ such that $\rho(t) + \rho(z) = k$ and moreover $\rho(z + \lambda t) = k$ for all $\lambda \neq 0, \lambda \in F$. Let

$$Tz = \sum_{i=1}^{m} u_i \otimes v_i , \qquad m = \rho(Tz) .$$

Since $u_i \in U$ are linearly independent and also $v_i \in V$ we can consider them as contained in a basis of U and V, respectively. The matrix of coordinates of Tz has the form

$$\left(\begin{array}{c|c} I_m & 0 \\ \hline 0 & 0 \end{array}\right)$$

where I_m is the identity $m \times m$ matrix. Let

$$\left(\begin{array}{c|c} A_m & B \\ \hline C & D \end{array}\right)$$

be the matrix of coordinates of Tt. Then the minor $|I_m + \lambda A_m|$ of the matrix of $T(z + \lambda t)$ has the form