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AN EXPLICIT FORMULA FOR THE UNITS OF AN ALGEBRAIC NUMBER FIELD OF DEGREE $n \geqq 2$

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An infinite set of algebraic number fields is constructed; they are generated by a real algebraic irrational w, which is the root of an equation $f(w) = 0$ with integer rational coefficients of degree $n \geqq 2$. In such fields polynomials $P_s(w) = a_0w^s + a_1w^{s-1} + \cdots + a_{s-1}w + a_s$ and

$$Q_s(w) = b_0w^s + b_1w^{s-1} + \cdots + b_{s-1}w + b_s$$

($s = 1, \cdots, n-1$; a_k, b_k rational integers) are selected so that the Jacobi-Perron algorithm of the $n - 1$ numbers

$$P_{n-1}(w), P_{n-2}(w), \cdots, P_1(w)$$

carried out in this decreasing order of the polynomials, and of the $n - 1$ numbers

$$Q_1(w), Q_2(w), \cdots, Q_{n-1}(w)$$

carried out in this increasing order of the polynomials both become periodic.

It is further shown that $n - 1$ different Modified Algorithms of Jacobi-Perron, each carried out with $n - 1$ polynomials $P_{n-1}(w), P_{n-2}(w), \cdots, P_1(w)$ yield periodicity. From each of these algorithms a unit of the field $K(w)$ is obtained by means of a formula proved by the authors is a previous paper.

It is proved that the equation $f(x) = 0$ has n real roots when certain restrictions are put on its coefficients and that, under further restrictions, the polynomial $f(x)$ is irreducible in the field of rational numbers. In the field $K(w)$ $n - 1$ different units are constructed in a most simple form as polynomials in w; it is proved in the Appendix that they are independent; the authors conjecture that these $n - 1$ independent units are basic units in $K(w)$.

1. Algorithm of $n - 1$ numbers. An ordered $(n - 1)$-tuple

$$(a_1^{(0)}, a_2^{(0)}, \cdots, a_{n-1}^{(0)}) , \qquad (n \geqq 2) \tag{1}$$

of given numbers, real or complex, among whom there is at least one irrational, will be called a basic sequence; the infinitely many $(n - 1)$-tuples

$$(b_1^{(v)}, b_2^{(v)}, \cdots, b_{n-1}^{(v)}) , \qquad (v = 0, 1, \cdots) \tag{2}$$

will be called supporting sequences. We shall denote by