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ON GENERAL Z.P.I.-RINGS

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A commutative ring in which each ideal can be expressed as a finite product of prime ideals is called a *general Z.P.I.-ring* (for Zerlegungsatz in Primideale). A general Z.P.I.-ring in which each proper ideal can be uniquely expressed as a finite product of prime ideals is called a *Z.P.I.-ring*. Such rings occupy a central position in multiplicative ideal theory. In case R is a domain with identity, it is clear that R is a Dedekind domain[1] and the ideal theory of R is well known. If R is a domain without identity, the following result of Gilmer gives a somewhat less known characterization of R: If D is an integral domain without identity in which each ideal is a finite product of prime ideals, then each nonzero ideal of D is principal and is a power of D; the converse also holds. Also somewhat less known is the characterization of a general Z.P.I.-ring with identity as a finite direct sum of Dedekind domains and special primary rings.[2]

This paper considers the one remaining case: R is a general Z.P.I.-ring with zero divisors and without identity. A characterization of such rings is given in Theorem 2. This result is already contained ih a more obscure form in a paper by S. Mori. The main contribution here is in the directness of the approach as contrasted to that of Mori.

In order to prove Theorem 2 we need to establish two basic properties of a general Z.P.I.-ring R: R is Noetherian and primary ideals of R are prime powers. Having established these two properties of R, the following result of Butts and Gilmer in [3], which we label as (BG), is applicable and easily yields our characterization of general Z.P.I.-rings without identity.

(BG), [3; Ths. 13 and 14]: *If R is a commutative ring such that $R \neq R^2$ and such that every ideal in R is an intersection of a finite number of prime power ideals, then $R = F_1 \oplus \cdots \oplus F_k \oplus T$ where each F_i is a field and T is a nonzero ring without identity in which every nonzero ideal is a power of T.*

It is important to note that we do not use Butts and Gilmer's

[1] M. Sono [**14**] and E. Noether [**13**] were among the first to consider Dedekind domains. For a historical development of the theory of Dedekind domains see [**4**; pp. 31-32].

[2] S. Mori in [**11**] considered both general Z.P.I.-rings with identity and Z.P.I.-rings without identity which contain no proper zero divisors, but Mori's results in these cases are not as sharp as those of Asano and Gilmer.