PACIFIC JOURNAL OF MATHEMATICS
Vol. 30, No. 3, 1969

HOLOMORPHIC QUADRATIC DIFFERENTIALS ON SURFACES IN E^3

TILLA KLOTZ

Let R be a Riemann surface defined upon an oriented surface S smoothly immersed in E^3. This paper studies holomorphic quadratic differentials on R which are related to the geometry on S, especially those of the form

$$\Omega_{\hat{A}} = \{(A - C) - 2iB\}dz^2$$

where $\hat{A} = Adx^2 + 2Bdxdy + Cdy^2$ is a smooth linear combination $\hat{A} = \hat{f}I + \hat{g}II$ of the fundamental forms on S, and $z = x + iy$ is any conformal parameter on R. Most results deal with the case in which $R = R_A$ is determined on S by some smooth positive definite linear combination $A = fI + gII$ on S. It is shown, for example, that S is isothermal with respect to A if and only if R_A supports a holomorphic $\Omega_{\hat{A}} \not\equiv 0$ in some neighborhood of any nonumbilic point. By way of contrast, another result states that a holomorphic $\Omega_{\hat{A}} \not\equiv 0$ is automatically available in the neighborhood of any nonumbilic point p, unless R coincides at p with some R_A. The paper closes with a study of surfaces which support an R_A on which both $\Omega_I \not\equiv 0$ and $\Omega_{II} \not\equiv 0$ are holomorphic.

Suppose that S is an oriented surface which is smoothly immersed in E^3, and that R is a Riemann surface defined upon the underlying 2-manifold of S so that each conformal parameter $z = x + iy$ on R yields a smooth, properly oriented coordinate pair x, y on S. As an example, $R = R_\theta$ might be determined on S by some sufficiently smooth positive definite quadratic form θ (see [1], § 4).

Ordinarily, there is no reason to expect any special relationship between the holomorphic quadratic differentials on R and the differential geometry on S. In particular, if one takes some quadratic form $\hat{\theta}$ of geometric interest on S, and associates with $\hat{\theta}$ the quadratic differential $\Omega_{\hat{\theta}} = \varphi_{\hat{\theta}} dz^2$ on R where $\hat{\theta} = Adx^2 + 2Bdxdy + Cdy^2$ and $\varphi_{\hat{\theta}} = (A - C) - 2iB$, then $\Omega_{\hat{\theta}}$ will not usually be holomorphic, that is, $\varphi_{\hat{\theta}}$ will not in general be analytic as a function of the conformal parameter $z = x + iy$ on R. There is always, of course, the trivial situation in which $\hat{\theta}$ is proportional to θ on S, so that $\Omega_{\hat{\theta}} \equiv 0$ is automatically holomorphic on R_θ. Yet in a striking number of cases, surfaces of particular interest to differential geometers have been shown to support a nontrivial holomorphic quadratic differential $\Omega_{\hat{\theta}} \not\equiv 0$ on some specific R. It seems appropriate to note a few such examples, as they provided the major motivation for the study undertaken in this paper.