PACIFIC JOURNAL OF MATHEMATICS
Vol. 31, No. 1, 1969

ISOMETRIES OF CERTAIN FUNCTION SPACES

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Let X be a discrete symmetric Banach function space with absolutely continuous norm. We prove by the method of generalized hermitian operator that an operator U on X is an onto isometry if and only if it is of the form:

$$Uf(.)=u(.)f(T.) \quad \text{all} \quad f \in X\,,$$

where u is a unimodular function and T is a set isomorphism of the underlying measure space. That other types of isometries occur if the symmetry condition is not present is illustrated by an example. We completely describe the isometries of a reflexive Orlicz space $L_{M\Phi}(\neq L_2)$ provided the atoms have equal mass (the atom-free case has been treated by G. Lumer); similarly for the case that no Hilbert subspace occurs.

We shall reproduce some definitions and results from [4] which will be needed in the sequel.

DEFINITION. Let X be a vector space. A semi-inner-product on X is a mapping $[\,,\,]$ of $X \times X$ into the field of numbers (real or complex) such that

$$\begin{aligned} [x + y, z] &= [x, z] + [y, z] \\ \lambda[x, z] &= [\lambda x, z] \text{ for all } x, y, z \in X \text{ and } \lambda \text{ scaler}\,. \\ [x, x] &> 0 \text{ for all } x \neq 0 \\ |[x, y]|^2 &\leqq [x, x][y, y]\,. \end{aligned}$$

We call X a semi-inner-product space (in short, s.i.p.s.). If X is a s.i.p.s., one shows easily that $[x, x]^{1/2}$ is a norm on X. On the other hand, let X be a normed space and X^* its dual. For each $x \in X$, there exists by the Hahn-Banach theorem, at least one (and we shall choose one) functional $Wx \in X^*$ such that $\langle x, Wx\rangle = \|x\|^2$. Given any such mapping W from X into X^* (ank in general, there are infinitely many such mappings), it is at once verified that $[x, y] = \langle x, Wy\rangle$ defines a semi-inner-product (s.i.p.).

DEFINITION. Given a linear transformation T on a s.i.p.s., we call the set $W(T) = \{[Tx, x]: [x, x] = 1\}$ the numerical range of T.

An important fact concerning the notion of numerical range is the following [4, Th. 14]:

Let X be a complex Banach space, and T an operator on X.