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COHESIVE SETS AND RECURSIVELY ENUMERABLE DEDEKIND CUTS

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In this paper the methods of recursive function theory are applied to certain classes of real numbers as determined by their Dedekind cuts or by their binary expansions. Instead of considering recursive real numbers as in constructive analysis, we examine real numbers whose lower Dedekind cut is a recursively enumerable (r.e.) set of rationals, since the r.e. sets constitute the most elementary nontrivial class which includes nonrecursive sets. The principal result is that the sets A of natural numbers which "determine" such real numbers α (in the sense that the characteristic function of A corresponds to the binary expansion of α) may be very far from being r.e., and may even be cohesive. This contrasts to the case of recursive real numbers, where A is recursive if and only if the corresponding lower Dedekind cut is recursive.

With each subset A of the set of natural numbers N, there is naturally associated a real number in the interval $[0, 2]$, namely $\Phi(A) = \sum_{n \in A} 2^{-n}$, and $\Phi(\emptyset) = 0$. Fix a one-one effective map from N onto Q, the set of rationals in the interval $[0, 2]$, and denote the image under this map of an element n by the **bold** face $\mathbf{n}$. Identifying each natural number n with its rational image $\mathbf{n}$, the (lower) *Dedekind cut associated with* A is simply

$$L(A) = \{n \mid \mathbf{n} \leqq \Phi(A)\} .$$

It is well known in recursive analysis [4] that A is recursive if and only if $L(A)$ is recursive, and in this case $\Phi(A)$ is said to be a *recursive real number*.

From the point of view of recursion theory, however, it is more natural to consider certain wider classes of Dedekind cuts, especially those which are recursively enumerable (r.e.). The most interesting results in recursion theory concern these sets. In going from recursive to recursively enumerable Dedekind cuts, we find that: A r.e. implies $L(A)$ r.e.; but not conversely. (C.G. Jockusch has observed the following simple counter-example to the converse. If A is any r.e. set and if $B = A$ join $\bar{A} = \{2n \mid n \in A\} \cup \{2n + 1 \mid n \in \bar{A}\}$, then $L(B)$ is r.e., but B is not r.e. unless A is recursive.) It is now natural to ask just how "sparse" the set A can be so that $L(A)$ remains r.e. At the end of §3 in [8] we indicated how to construct a hyperimmune set H such that $L(H)$ is r.e. We now consider two notions (dominant and hyper-