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APPROXIMATION BY INNER FUNCTIONS

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Let $L^\infty(T)$ denote the complex Banach algebra of (equivalence classes of) bounded measurable functions on the unit circle T, relative to Lebesgue measure m. The norm $\|f\|_\infty$ of an f in $L^\infty(T)$ is the essential supremum of $|f|$ on T. The collection of all bounded holomorphic functions in the open unit disc U forms a Banach algebra which can be identified (via radial limits) with the norm-closed subalgebra H^∞ of $L^\infty(T)$.

A function f in $L^\infty(T)$ is unimodular if $|f| = 1$ a.e., on T. The inner functions are the unimodular members of H^∞. It is well known that they play an important role in the study of H^∞.

The main result (Theorem 1) is that the set of quotients of inner functions is norm-dense in the set of unimodular functions in $L^\infty(T)$. One consequence of this (Theorem 7) is that the set of radial limits of holomorphic functions of bounded characteristic in U is norm-dense in $L^\infty(T)$. It is also shown (Theorem 3, 4) that the Gelfand transforms of the inner functions separate points on the Šilov boundary of H^∞, and this is used to obtain a new proof (and generalization) of a theorem of D. J. Newman (Theorem 4).

Our proof of the main result uses only one nontrivial property of H^∞, beyond the fact that H^∞ is a norm-closed subalgebra of L^∞. It therefore applies, without any extra effort, to a much more general situation which we now describe.

Let now L^∞ denote the Banach algebra of all bounded measurable functions on some measure space X, normed by the essential supremum, and let B be a norm-closed subalgebra of L^∞. We say that B has the *annulus property* if the following is true:

If X is the union of disjoint measurable sets E_1 and E_2 and if $0 < r_1 < r_2 < \infty$, then there exists h in B such that

(1) *$1/h$ is in B, and*

(2) $|h| = r_i$ a.e., *on E_i, for $i = 1, 2$.*

That H^∞ (in the classical setting described above) has the annulus property is well known: to see it, put $u = r_i$ on E_i (now $T = E_1 \cup E_2$), and define

$$h(z) = \exp\left\{\frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{e^{i\theta} + z}{e^{i\theta} - z} \log u(e^{i\theta})d\theta\right\} \qquad (z \in U).$$

Then h maps U into the annulus $\{w: r_1 < |w| < r_2\}$, and the radial limits of h have modulus r_i a.e., on E_i.