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THE NORM OF A DERIVATION

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In this paper, we determine the norm of the inner derivation $\mathfrak{D}_T$: $A \to TA - AT$ acting on the Banach algebra $\mathfrak{B}(H)$ of all bounded linear operators on Hilbert space. More precisely, we show that $\|\mathfrak{D}_T\| = \inf\{2\|T - \lambda I\|: \lambda \text{ complex}\}$. If T is normal, then $\|\mathfrak{D}_T\|$ can be specified in terms of the geometry of the spectrum of T.

A derivation on a Banach algebra $\mathfrak{A}$ is a linear transformation $\mathfrak{D}: \mathfrak{A} \to \mathfrak{A}$ which satisfies $\mathfrak{D}(ab) = a\mathfrak{D}(b) + \mathfrak{D}(a)b$ for all $a, b \in \mathfrak{A}$. If for a fixed a, $\mathfrak{D}: b \to ab - ba$, then $\mathfrak{D}$ is called an inner derivation. Sakai has shown that every derivation on a von Neumann algebra [8] or on a simple C^*-algebra [9] is inner. See also [3] and [4].

In [7], Rosenblum determined the spectrum of an inner derivation. Our estimates on the norm of $\mathfrak{D}_T$ have some applications of general operator theory as a by product. Kadison, Lance, and Ringrose [5] have investigated the derivation $\mathfrak{D}_T$ acting on a general C^*-algebra, when T is self adjoint. In §2, we study $\mathfrak{D}_T$ acting on an irreducible C^*-algebra. There appears to be little common ground in the two approaches. In the last section we consider the operator which sends $X \to AX - XB$ for $A, B, X \in \mathfrak{B}(H)$.

1. From now on, all operators are bounded and act on a Hilbert space. We shall denote the complex numbers by $\boldsymbol{C}$.

DEFINITION. The maximal numerical range of T is the set

$$W_0(T) = \{\lambda : (Tx_n, x_n) \to \lambda \text{ where } \|x_n\| = 1 \text{ and } \|Tx_n\| \to \|T\|\}.$$

When H is finite dimensional, $W_0(T)$ corresponds to the numerical range produced by the maximal vectors (vectors x such that $\|x\| = 1$ and $\|Tx\| = \|T\|$).

LEMMA 1. *If* $\|T\| = \|x\| = 1$ *and* $\|Tx\|^2 \geqq (1 - \varepsilon)$, *then* $\|(T^*T - I)x\|^2 \leqq 2\varepsilon$.

Proof. Note that $0 \leqq \|(T^*T-I)x\|^2 = \|T^*Tx\| - 2\|Tx\|^2 + \|x\|^2 \leqq 2(1 - \|Tx\|^2) \leqq 2\varepsilon$.

LEMMA 2. *The set $W_0(T)$ is nonempty, closed, convex, and contained in the closure of the numerical range.*