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LOCALIZATION OF THE CORONA PROBLEM

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The corona problem for planar open sets D and the fibers of the maximal ideal space of $H^\infty(D)$ are discussed and shown to depend only on the local behavior of D.

Let D be an open subset of the Riemann sphere C^*, and let $H^\infty(D)$ be the uniform algebra of bounded analytic functions on D. We will assume always that $H^\infty(D)$ contains a nonconstant function, that is, that $C^*\backslash D$ has positive analytic capacity. Our object is to study the maximal ideal space $\mathscr{M}(D)$ of $H^\infty(D)$, and the "fibers" $\mathscr{M}_\lambda(D)$ of $\mathscr{M}(D)$ over points $\lambda \in \partial D$. The basis for our investigation is the observation that the fiber $\mathscr{M}_\lambda(D)$ depends only on the behavior of D near λ. This localization principle is used to obtain information related to the corona problem.

The corona of D is the part of $\mathscr{M}(D)$ which does not lie in the closure of D. Our main positive results are that D has no corona under either of the following assumptions:

(1) that the diameters of the components of $C^*\backslash D$ (in the spherical metric, if D is unbounded) be bounded away from zero; or

(2) that for some fixed $m \geqq 0$, the complement of each component of D has $\leqq m$ components.

The proofs rest on the localization principle, and on Carleson's solution of the corona problem for the open unit disc [2]. Each of the above conditions includes the extension of Carleson's theorem to finitely connected planar domains due to Stout [9].

In the negative direction, we present an example, due to E. Bishop, of a connected one-dimensional analytic variety W which is not dense in the maximal space of $H^\infty(W)$. The construction is similar to that of Rosay [8].

1. Two basic lemmas. The localization process depends on the following two lemmas.

LEMMA 1.1. *Let $\lambda \in \partial D$, and let U be an open neighborhood of λ. If $f \in H^\infty(D \cap U)$, there is $F \in H^\infty(D)$ such that $F - f$ extends to be analytic at λ, and $(F - f)(\lambda) = 0$. Moreover, F can be chosen so that $||F||_D \leqq 33||f||_{D\cap U}$.*

Indication of proof. Suppose $U = \Delta(\lambda; \delta)$ is the disc of radius δ, centered at λ. Let g be a smooth function supported on U, such that $g = 1$ on $\Delta(\lambda; \delta/2)$, and $|\partial g/\partial \bar{z}| \leqq 4/\delta$. Define $f = 0$ off D, and set