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ON THE NUMBER OF NONPIERCING POINTS IN CERTAIN CRUMPLED CUBES

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Let K denote the closure of the interior of a 2-sphere S topologically embedded in Euclidean 3-space E^3. If $K - S$ is an open 3-cell, McMillan has proved that K has at most one nonpiercing point. In this paper we use a more general condition restricting the complications of $K - S$ to describe the number of nonpiercing points. The condition is this: for some fixed integer n $K - S$ is the monotone union of cubes with n holes. Under this hypothesis we find that K has at most n nonpiercing points (Theorem 5). In addition, the complications of $K - S$ are induced just by these nonpiercing points. Generally, at least two such points are required, for otherwise $n = 0$ (Theorem 3).

A space K as described above is called a *crumpled cube*. The *boundary of* K, denoted $\operatorname{Bd} K$, is defined by $\operatorname{Bd} K = S$, and the *interior of* K, denoted $\operatorname{Int} K$, is defined by $\operatorname{Int} K = K - \operatorname{Bd} K$. We also use the symbol Bd in another sense: if M is a manifold with boundary, then $\operatorname{Bd} M$ denotes the boundary of M. This should not produce any confusion.

Let K be a crumpled cube and p a point in $\operatorname{Bd} K$. Then p is a *piercing point of* K if there exists an embedding f of K in the 3-sphere S^3 such that $f(\operatorname{Bd} K)$ can be pierced with a tame arc at $f(p)$.

Let U be an open subset of S^3. The *limiting genus of* U, denoted $LG(U)$, is the least nonnegative integer n such that there exists a sequence $H_1, H_2, \cdots$ of compact 3-manifolds with boundary satisfying (1) $U = \cup H_i$, (2) $H_i \subset \operatorname{Int} H_{i+1}$, and (3) genus $\operatorname{Bd} H_i = n$ ($i = 1, 2, \cdots$). If no such integer exists, $\operatorname{LG}(U)$ is said to be infinite. Throughout this paper the manifolds H_i described above can be obtained with connected boundary, in which case H_i is called a *cube with n holes*.

Applications of the finite limiting genus condition are investigated in [6] and [14]. *For any crumpled cube K such that* $\operatorname{LG}(\operatorname{Int} K)$ is finite and $\operatorname{Bd} K$ is locally peripherally collared from $\operatorname{Int} K$, it is shown that $\operatorname{Bd} K$ is locally tame (from $\operatorname{Int} K$) except at a finite set of points. Under the hypothesis of this paper, $\operatorname{Bd} K$ may be wild at every point; nevertheless, with a collapsing (in the sense of Whitehead [15]) argument comparable to [13, Th. 1], the problem of counting the nonpiercing points of K is reduced to one in which the results of [6] and [14] apply.

A subset X of the boundary of a crumpled cube K is said to be *semi-cellular in* K if for each open set U containing X there exists