PACIFIC JOURNAL OF MATHEMATICS
Vol. 34, No. 2, 1970

A DENSITY WHICH COUNTS MULTIPLICITY

ROBERT E. DRESSLER

P. Erdös, using analytic theorems, has proven the following results: Let $f(x)$ be the number of integers m such that $\phi(m) \leqq x$, where ϕ is the Euler function, and let $g(x)$ be the number of integers n such that $\sigma(n) \leqq x$, where σ is the usual sum of divisors function. Then there are positive (but undetermined) constants c_1 and c_2 such that $f(x) = c_1 x + o(x)$ and $g(x) = c_2(x) + o(x)$. The constants c_1 and c_2 can be calculated using complex analysis including the Wiener-Ikehara Theorem. A major purpose of this paper is to give an elementary proof that $\lim_{x\to\infty} f(x)/x$ exists and, in the process, calculate the value of the limit. These considerations of multiplicity motivate a generalization of natural density which counts multiplicity. This paper contains an investigation of this generalization.

Let $A = \{a_i\}_{i=1}^{\infty}$ be a sequence of positive real numbers $\geqq 1$. For a positive integer j, define $\#(A, j)$ to be the number of integers i such that $a_i \leqq j$ (that is, the number of elements of A *counting multiplicity* which are $\leqq j$). If $\liminf_{j\to\infty} \#(A, j)/j = \alpha$ (we allow $\alpha = \infty$) we say A has Δ-asymptotic density α and we define $\underline{\Delta}(A) = \alpha$. We also define $\bar{\Delta}(A) = \limsup_{j\to\infty} \#(A, j)/j$. If $\underline{\Delta}(A) = \bar{\Delta}(A)$ we say A has Δ-natural density α and we define $\Delta(A) = \alpha$. It is clear that a reordering of A does not affect $\underline{\Delta}(A)$ or $\bar{\Delta}(A)$. It is also clear that $\underline{\Delta}(A) = \underline{\Delta}(\{[a_i]\}_{i=1}^{\infty})$ and $\bar{\Delta}(A) = \bar{\Delta}(\{[a_i]\}_{i=1}^{\infty})$ where $[a_i]$ is the greatest integer which does not exceed a_i. Unless otherwise specified all sequences in this paper will be of positive real numbers.

Throughout this paper d will denote natural density, i.e., the classical analog of Δ where multiplicity is not counted; Z^+ will denote the set of positive integers; Q^+ will denote the positive rational numbers; R^+ will denote the set of positive real numbers; p will always be a prime; and $P = \{p_i\}_{i=1}^{\infty}$ will be the sequence, in the natural order, of primes.

If γ: $Z^+ \rightarrow R^+$ then to γ there corresponds the unique sequence $\gamma(1), \gamma(2), \cdots$. We will write γ in place of this sequence. Thus, for example, in the notation of this paper $\Delta(\phi)$ and $\Delta(\sigma)$ exist and are positive [5]. If for instance $\gamma = \tau$, where $\tau(n) =$ the number of positive integer divisors of the positive integer n, then it is clear that $\Delta(\tau) = \infty$.

If $A = \{a_i\}_{i=1}^{\infty}$ and $B = \{b_j\}_{j=1}^{\infty}$ are sequences then define $A + B$ to be the sequence, in the natural order, of positive real numbers x such that there exist i and $j \in Z^+$ with $a_i + b_j = x$, and x appears in this