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FUNCTION SPACE TOPOLOGIES

J. D. HANSARD

S. Naimpally [3] introduced the graph topology, Γ, for function spaces. H. Poppe [5] showed that if the graph topology is finer than the topology of uniform convergence, τ_u, or finer than the finest of the σ-topologies of Arens and Dugundji, τ, and if the range space is the real line, R, then the domain is countably compact.

We assume our range space is R and that our domain space X is T_1. In most of this paper we deal with topologies on $C(X)$ the set of continuous real-valued functions on X. We show that $\Gamma = \tau = \tau_u$ on $C(X)$ if and only if X is countably compact. Further, we find that when X is locally connected, $\tau_u \subset \tau$ on $C(X)$ if and only if X has finitely many components.

In order to determine conditions under which $\tau \subset \tau_u$, we introduce a map extension property between complete regularity and normality and show that for domain spaces X having this property, $\tau \subset \tau_u$ on $C(X)$ if and only if X is countably compact. We indicate further applications of this map extension property and compare it to weak normality.

We let Y^X denote the set of all functions from X to Y. For $f \in Y^X$, let $G(f) = \{(x, f(x)) : x \in X\}$, and for $U \subset X \times Y$, let $F_U = \{f \in Y^X : G(f) \subset U\}$. If $K \subset X$ and $U \subset Y$, define $(K, U) = \{f \in Y^X : f(K) \subset U\}$. For $\varepsilon > 0$, let $V_\varepsilon(f) = \{g \in R^X : |f(x) - g(x)| < \varepsilon$ for all $x \in X\}$. Also, define $N_\varepsilon(x) = \{y \in R : |y - x| < \varepsilon$, and for any set K, let cK be the complement of K.

The graph topology, defined in [3], has a basis consisting of the sets of the form F_U where U is open in $X \times Y$. The finest of the σ-topologies, defined in [1], has a subbasis consisting of the collection of all sets of the form (K, U) where $K \subset X$ is closed and $U \subset Y$ is open. The topology of uniform convergence has a basis consisting of all sets of the form $V_\varepsilon(f)$ where $f \in R^X$ and $\varepsilon > 0$.

1. Two lemmas.

The first of our lemmas is a characterization of τ which we find convenient to use throughout this paper. This result provides us with a basis for τ. Because of the nature of these basic elements, the relation between τ and Γ is immediately made clear, and we are able to think of τ, intuitively, as a special kind of graph topology rather than as a set-open topology.

H. Poppe [5] showed that τ has a subbasis consisting of sets of the form $[K \times L] = \{f : G(f) \cap K \times L = \varnothing$, $K \subset X$ closed, $L \subset Y$ closed$\}$. Thus a basic open set of functions in τ, $\bigcap_1^n [K_i \times L_i]$, is completely