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COUNTABLE BOOLEAN ALGEBRAS AS SUBALGEBRAS AND HOMOMORPHS

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The problem of classifying all countable Boolean algebras appears to be impossible to solve. This paper considers the following problem. Given a class $\mathscr{C}$ of countable Boolean algebras, which is closed under isomorphisms, characterize the classes of

(i) all Boolean algebras which have subalgebras in $\mathscr{C}$;

(ii) all subalgebras af members of $\mathscr{C}$;

(iii) all homomorphs of members of $\mathscr{C}$;

(iv) all Boolean algebras which have homomorphs in $\mathscr{C}$.

Definitive characterizations are obtained for the first three classes (Theorems 7, 8, and 9), and a representation of the last class is obtained when $\mathscr{C}$ is the class of all countable Boolean algebras (Theorem 11).

Given a Boolean algebra A, let X be the corresponding Boolean space. Define inductively: $D_0(X)$ as X; $D_1(X)$ as the complement of the isolated points in X; for any ordinal α, $D_{\alpha+1}(X)$ as $D_1(D_\alpha(X))$; and for a limit ordinal α, $D_\alpha(X)$ as $\cap\{D_\beta(X)\colon \beta < \alpha\}$. Then $D_\alpha(X)$ is a closed subspace of X for each ordinal α. The Boolean algebra A is said to be *superatomic* if $D_\alpha(X) = \varnothing$, for some ordinal α. If γ is the last ordinal for which $D_\gamma(X) = \varnothing$, the *cardinal sequence* $\Gamma(A)$ of the superatomic Boolean algebra A is defined in [1] as the sequence of order type γ whose α-term is the cardinality of the set of isolated points of $D_\alpha(X)$, $\alpha < \gamma$. Note that each term of $\Gamma(A)$ is infinite except for the $\gamma - 1$ term, which must be finite.

If $\mathscr{S}$ is the class of all superatomic Boolean algebras, the class $\{\Gamma(A)\colon A \in \mathscr{S}\}$ may be partially ordered by setting $\Gamma(A) \leqq \Gamma(B)$ when the order type of $\Gamma(A)$ is not larger than that of $\Gamma(B)$, and when the α-term of $\Gamma(A)$ is not larger than the α-term of $\Gamma(B)$. It is shown in [4] that every countable superatomic Boolean algebra has as its Boolean space the ordinal $\omega^\alpha n + 1$ endowed with the order topology, for some countable ordinal α and natural numbers n. Thus the relation "$\leqq$" well orders the class of cardinal sequences of countable superatomic Boolean algebras.

LEMMA 1. *If X and Y are Boolean spaces, and θ is a continuous function from X onto Y, then $\theta[D_\alpha(X)]$ contains $D_\alpha(Y)$, for each ordinal α.*

Proof. Assume that $\alpha = 1$, and that y is an element of $Y -$