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A GENERALIZATION OF MARTINGALES AND TWO CONSEQUENT CONVERGENCE THEOREMS

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Loeve has observed that a discrete stochastic process can be interpreted as a game and that a martingale can be interpreted as a "fair" game. In this context, the notion of a martingale is enlarged to a game which becomes "fairer with time" and then this concept is utilized to establish two convergence theorems.

Let $(\Omega, \mathfrak{A}, p)$ be a probability space with $\{\mathfrak{A}_n\}_{n\geq 1}$ an increasing family of sub σ-algebras of $\mathfrak{A}$ to which the process $\{X_n\}_{n\geq 1}$ is adapted, (see [3, p. 65]). Henceforth, the process $\{X_n\}_{n\geq 1}$ will be referred to as a game.

DEFINITION. The game $\{X_n\}_{n\geq 1}$ will be said to become *fairer with time* if for every $\varepsilon > 0$.

$$p[|E(X_n \mid \mathfrak{A}_m) - X_m| > \varepsilon] \to 0$$

as $n, m \to \infty$ with $n \geqq m$.

It should be noted that any martingale is a game which becomes fairer with time. An easy example of a game which is not a martingale or a sub or a super martingale but does become fairer with time is constructed by considering a game which consists of tossing a die. Here, let

$$\mathfrak{A}_n = \mathfrak{A}, \text{ all } n$$

and

$$X_n(\{i\}) \equiv i + (-1)^n/n\ .$$

The main results. Let $\{\alpha_n : n \geqq 1\}$ be a monotonic sequence decreasing to zero with finite sum. The game $\{X_n\}_{n\geq 1}$ may be decomposed with respect to $\{\alpha_n : n \geqq 1\}$ as

$$X_n = Y_n - Z_n, \text{ where } \{Y_n\}_{n\geq 1} \text{ and } \{Z_n\}_{n\geq 1} \tag{1.1}$$

are defined inductively by:

$$\begin{aligned} &Y_1 = X_1 \\ &\vdots \\ &Y_n = Y_{n-1} + [X_n - E(X_n \mid \mathfrak{A}_{n-1})] + \alpha_{n-1} \end{aligned} \tag{1.2}$$