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ON THE SEMIGROUP OF BINARY RELATIONS

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The concepts of row and column bases for an element of $\mathscr{B}_X$, the semigroup of binary relations on a set X, are introduced by interpreting a binary relation as a boolean matrix; these ideas are then used to characterize the Green's equivalences on $\mathscr{B}_X$. It is shown that the class of idempotent relations whose rows and columns form independent sets coincides with the class of partial order relations on subsets of X. Regularity in $\mathscr{B}_X$ is investigated using these results.

The Green's relations and ideals in the semigroup of binary relations $\mathscr{B}_X$ on a set X have been studied primarily in terms of lattice considerations [11], [12]. In this paper we take a more computational approach. By interpreting a relation as a boolean matrix, we introduce the concept of row and column bases and use these ideas to obtain useful characterizations of the Green's relations $\mathscr{L}$, $\mathscr{R}$, $\mathscr{H}$ and $\mathscr{D}$ on $\mathscr{B}_X$. These results are then used to investigate the ideal structure of $\mathscr{B}_X$, in comparison to that of $\mathscr{T}_X$, the semigroup of transformations of X into X. Some simple tests for regularity of a binary relation are obtained, and by characterizing reduced idempotent relations we show that a regular relation must have the same row rank and column rank.

These results have made possible the determination of the maximal subgroups of $\mathscr{B}_X$ [6]. Moreover, the characterization of the Green's relations in terms of binary matrices will hopefully lead to an extension of the combinatorial results given in [4] and [9], in which the numbers of idempotents in the $\mathscr{L}$ and $\mathscr{H}$-classes of $\mathscr{T}_X$ are investigated. Other applications may be found in Grapy Theory.

A binary relation on a set X is a subset of $X \times X$, and the set of all binary relations on X is denoted by $\mathscr{B}_X$. The product $\alpha\beta$ of two relations α and β on X is defined to be the relation

$$\alpha\beta = \{(a, b) \mid (a, c) \in \alpha \text{ and } (c, b) \in \beta \text{ for some } c \in X\} .$$

The operation is associative and hence $\mathscr{B}_X$ is a semigroup. The semigroup $\mathscr{P}_X$ of partial transformations on X is a subsemigroup of $\mathscr{B}_X$ and it in turn contains $\mathscr{T}_X$, the semigroup of transformations on X as a subsemigroup. It was the ideal structure of $\mathscr{T}_X$ that motivated many of the ideas in this paper. (See [5] and [1] Vol. I, pp. 51-55.)