PACIFIC JOURNAL OF MATHEMATICS
Vol. 37, No. 1, 1971

THE TRANSCENDENTAL RANK OF A THEORY

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Morley has associated with each countable complete theory T an ordinal $\alpha_T < (2^{\aleph_0})^+$. It is shown that in fact $\alpha_T \leqq \omega_1$ and that this bound is best possible.

We shall use the notation and terminology of Morley [1], where α_T is defined to be the least ordinal α such that for all $A \in N(T)$ and all $\beta > \alpha$, $S^\alpha(A) = S^\beta(A)$. As in [1] T denotes a complete theory in a countable language L, T has an infinite model, and there is a theory Σ such that $T = \Sigma^*$. If $A \in N(T)$ and $p \in S(A)$, let $r(p) = \alpha$ if p is transcendental in rank α and let $r(p)$ be undefined otherwise. Also, if $A \in N(T)$ and $\psi \in F(A)$ define

$$r(\psi, A) = \begin{cases} -1 & \text{if} \quad U_\psi = \varnothing \\ \sup\{\alpha \mid p \in U_\psi \,\&\, r(p) = \alpha\} & \text{otherwise.} \end{cases}$$

LEMMA. *Let $A \in N(T)$, $\psi \in F(A)$, and $r(\psi, A) = \alpha$. Then for each $\beta < \alpha$ there exists $B \in N(T)$, $A \subseteqq B$, and $q \in S(B)$ such that $r(q) = \beta$ and $\psi \in q$.*

Proof. Assume the hypothesis and for contradiction that no B and q exist satisfying the conclusion. Then for every $B \in N(T)$, $A \subseteqq B$, we have $i_{AB}^{*-1}(U_\psi) \cap Tr^\beta(B) = \varnothing$. Thus for all such B, $i_{AB}^{*-1}(U_\psi) \cap (S^{\beta+1})(B) - S^\beta(B)) = \varnothing$. Suppose $q' \in Tr^{\beta+1}(B)$ then for every $C \in N(T)$, $B \subseteqq C$, $i_{BC}^{*-1}(q') \cap S^{\beta+1}(C)$ is a set of isolated points in $S^{\beta+1}(C)$. Thus, if $\psi \in q'$, $i_{BC}^{*-1}(q') \cap S^\beta(C)$ is a set of isolated points in $S^\beta(C)$ for all such C, whence $q' \in Tr^\beta(B)$. We conclude that $i_{AB}^{*-1}(U_\psi) \cap Tr^{\beta+1}(B) = \varnothing$ for all $B \in N(T)$, $A \subseteqq B$. By induction $i_{AB}^{*-1}(U_\psi) \cap Tr^\gamma(B) = \varnothing$ for all $\gamma \geqq \beta$. This contradicts the hypothesis and completes the proof of the lemma.

From 2.3(b) and 2.4 of [1] it is possible to choose B in the conclusion of the lemma such that $\kappa(B - A) = \aleph_0$; we shall make use of this fact below.

Before proceeding further we need some more definitions. A language L_1 is said to be a *simple extension* of a language L_0 if it is obtained by adjoining $\aleph_0$ individual constants to L_0. For any language L' let $F(L')$ denote the set of formulas of L' which have no free variable other than v_0. For each $n \in \omega$ let S_n denote the set of all sequences of 0's and 1's of length $\leqq n$; the empty sequence $\varnothing$ is allowed. For $s \in S_n$ and $i \leqq 1$, $s * \langle i \rangle$ denotes the member of S_{n+1}