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ON THE LIMITING DISTRIBUTION OF ADDITIVE FUNCTIONS (MOD 1)

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A function $f(n)$, defined on the positive rational integers, is said to be additive if and only if for every pair of coprime integers a and b the relation

$$f(ab) = f(a) + f(b)$$

is satisfied. Thus an additive function is determined by its values on those integers which are prime powers. In an extensive paper Erdos raised the question of characterising those real valued additive functions which have a limiting distribution (mod 1).

It is our present purpose to give such a characterisation.

He proved, in particular, that an additive function $f(n)$ is certainly uniformly distributed in the sense of Weyl if $f(p) \to 0$ as $p \to \infty$, and if the series

$$\sum \frac{f^2(p)}{p}$$

diverges.

For the remainder of this paper we understand a *distribution function* $F(z)$ (mod 1), or more shortly *a distribution function*, to have the properties

(i) *$F(z)$ is increasing in the wide sense*

(ii) *$F(z) = F(z+)$ for all values of z, that is $F(z)$ is right continuous.*

(iii) *$F(z) = 0$ if $z < 0$, and $= 1$ if $z \geqq 1$.*

We say that a sequence of distribution functions $F_n(z)$, $n = 1, 2, \cdots$ has a *limiting distribution* (*mod* 1) if and only if there exists a function $F(z)$, satisfying the above three conditions, so that at every pair of points of continuity (α, β) of $F(z)$, $0 < \alpha < \beta < 1$, we have

$$F_n(\beta) - F_n(\alpha) \to (F(\beta) - F(\alpha)) , \qquad (n \to \infty) .$$

We notice that in the range $0 < z < 1$ any such limiting distribution $F(z)$ is determined only up to an additive constant. When the function $F(z)$ is