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A NOTE ON THE LÖWNER DIFFERENTIAL EQUATIONS

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The object of the present note is to indicate a derivation of the Löwner differential equations [1] based on the derivation of an associated differential equation for Green's function of the variable region relative to the defining parameter. Decisive in our treatment is the use of a certain normalized minimal positive harmonic function on the variable region. In fact, our starting point was the feeling that the Poisson kernel asserted its presence so strongly in the Löwner differential equations that the concomitant presence of a normalized minimal positive harmonic function on the variable region should appear naturally in the study of the question. We shall see that this is the case. A technical advantage of the present approach is that the "tip" lemmas of the classical proof are dispensed with.

It would be of interest to see whether the indicated method, which is available for other families of harmonic functions monotone justifying in a parameter, has useful applications to the theory of harmonic functions.

2. Let γ be a Jordan arc with parametric domain $[0, T]$ such that $0 < |\gamma(t)| < 1$ for $0 \leqq t < T$ and $|\gamma(T)| = 1$. Let A_t denote the complement of the set $\gamma(\{t \leqq s < T\})$ with respect to the open unit disk, $0 \leqq t \leqq T$. Let g_t denote Green's function for A_t with pole at 0. The continuous dependence of g_t on the parameter t is an elementary matter (minimal property of Green's function, the Phragmén-Lindelöf boundary maximum principle). We let $\alpha(t)$ denote $\lim_{z\to 0}$ $[g_t(z) + \log|z|]$. We note that $\alpha\colon t \to \alpha(t)$ is an increasing continuous function which satisfies $\alpha(T) = 0$. We reparametrize γ, as in the original Löwner argument, by composing γ with

$$t \longrightarrow \operatorname{inv} \alpha[t + \alpha(0)],\ 0 \leqq t \leqq -\alpha(0),$$

so that for the new γ we have $T = -\alpha(0)$ and $\alpha(t) = \alpha(0) + t$. [The notation "inv" is used to denote the inverse of a univalent function.]

We let G be defined by

$$G(z, t) = g_t(z),\ z \in A_t,\ 0 \leqq t \leqq T.$$