PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 1, 1971

REVERSIBLE HOMEOMORPHISMS OF THE REAL LINE

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Let G be the group of germs of C^k local homeomorphisms of the real line which fix the origin and have nonzero derivative there. In this paper the possibility of factoring an element of G which is conjugate to its inverse into the product of two involutions is investigated. It is shown that it is always possible to do this in the analytic case and not always possible in the continuous case. In the intermediate cases several necessary and sufficient conditions are developed for determining whether or not such a factorization is possible. Included is a construction which allows one to determine an explicit factorization. Indication is given of the application of this material to the same problem in higher dimensions. This work is related to some material in Dynamics.

1. Introduction. If G is an abstract group an element $g \in G$ is called *reversible* in G if there exists an element $h \in G$ such that $hgh^{-1} = g^{-1}$. The product of two involutions is always reversible by an obvious argument. There arises the following question:

Question $\sharp$. If g is reversible in G can g be factored into the product of two involutions in G?

D. C. Lewis has decided this issue in the case $G = GL(n, \boldsymbol{C})$ affirmatively (Lewis [4]).

This paper concerns itself with the investigation of this question in the case where G is the group of germs of continuous or differentiable homeomorphisms of the real line.

Reversible transformations play a role in Dynamics. For further information on this connection see the references in Lewis [4].

2. Definitions and Notation. $C^k = \{F$: F is a local homeomorphism of a neighborhood of 0 in $\boldsymbol{R}$ to another such neighborhood which fixes the origin and is of class C^k on some neighborhood of 0, $F'(0) \neq 0$ if $k > 0$, for $0 \leqq k \leqq \infty$ or analytic for $k = \omega\}$. $T^k =$ {germs of elements of $\boldsymbol{C}^k$}

Let ϕ_k: $\boldsymbol{C}^k \rightarrow \boldsymbol{T}^k$ be the map which assigns to each element of $\boldsymbol{C}^k$ its germ in $\boldsymbol{T}^k$. The binary operation of (local) composition of mappings in $\boldsymbol{C}^k$ induces a group multiplication in $\boldsymbol{T}^k$. $\boldsymbol{T}^k$ will be viewed as a group with this structure henceforward.