PACIFIC JOURNAL OF MATHEMATICS
Vol. 39, No. 1, 1971

GERŠGORIN THEOREMS, REGULARITY THEOREMS, AND BOUNDS FOR DETERMINANTS OF PARTITIONED MATRICES
II
SOME DETERMINANTAL IDENTITIES

J. L. BRENNER

A square matrix $A = [a_{ij}]_1^n$ has dominant diagonal if $\forall_i\{|a_{ii}| > R_i = \sum_{j\neq i}|a_{ij}|\}$. A more complicated type of dominance is the following. Suppose for each i, there is assigned a set $I(i)$ (subset of $\{1, \cdots, n\}$), $i \in I(i)$: Define B_{ij} as the $I(i) \times I(i)$ submatrix of A that uses columns $I(i)$, and rows $\{I(i)\backslash i, j\}$, i.e., the set obtained from $I(i)$ by replacing the ith row by the jth row. Set $b_{ij} = \det B_{ij}$. Then $[b_{ij}]_1^n$ is a matrix, the elements of which are determinants of minor matrices of A. In an earlier paper, bounds for det A were derived in case $[b_{ij}]$ has dominant diagonal in the special case that $\{I(i)\}_i$ represents a partitioning of the indices into disjoint subsets.

In this article the general case is treated; $I(i)$ can be any subset of $\{1, \cdots, n\}$ that contains i. An identity is derived connecting $\det[b_{ij}]_1^n$ with $\det A$.

To establish the identity, a general multinomial identity is first derived, connecting determinants of certain submatrices of an $r \times 2r$ matrix of indeterminates. This result, reminiscent of Sylvester's determinantal identity, is used to bound $\det A$.

1. Application of a characterization of the determinant function.

LEMMA 1.01. *Let $A = [a_{ij}]_1^n$ be a matrix of complex numbers* [or indeterminates]; *let a function $\phi: A \rightarrow C$*[or $\phi: A \rightarrow C[a_{11}, \cdots, a_{nn}]$] *have the following properties for all $n \times n$ matrices A.*

(1.02) [1.03] *If any row* [*column*] *of A is replaced by the sum of that row* [*column*] *and a multiple of another row* [*column*], *$\phi(A)$ is unaltered.*

(1.04) *If any row of A is multiplied* (*throughout*) *by a constant α, $\phi(A)$ is multiplied α^r.*

Then $\phi(A)$ is a constant c_0 (*independent of a_{ij}*) *multiplied by the rth power of det A.*