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COMPLEX CHEBYSHEV ALTERATIONS

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P. Chebyshev's famous Alternation Theorem for best uniform approximation to continuous real valued functions on an interval is generalized to include best approximation to a class of continuous complex valued functions on an ellipse.

1. Preliminary remarks and definitions. For a continuous complex valued function f defined on a compact set E in the plane and, for $n \in Z^+$, let $p_n(f, E)$ denote the polynomial of degree n, of best uniform appoximation to f on E and *let*;

$$\rho_n(f, E) = \max_{z \in E} |f(z) - p_n(f, E)(z)| .$$

Chebyshev's Alternation Theorem [1, p. 29] states that if f is a continuous real valued function on an interval $[a, b]$, and p_n is a polynomial of degree n, $n \in Z^+$, then $p_n = p_n(f, [a, b])$ if and only if, there exists $n + 2$ points,

$\{x_i\}_{i=1}^{n+2}, a \leqq x_1 < x_2 < \cdots < x_{n+2} \leqq b$, with the property that $|f(x) - p_n(x)|$ attains its maximum on $[a, b]$ at these points and $f(x_i) - p_n(x_i) = -[f(x_{i+1}) - p_n(x_{i+1})]$ for $i = 1, 2, \cdots, n + 1$.

The sets we consider here are ellipses which are of course a generalization of intervals. So, for $a \geqq 0$, let $E_a = \{z + a/z: |z| = 1\}$. Now let $\mathscr{F}_n(E_n)$ denote those complex valued functions f, not themselves polynomials of degree n, continuous on E_a, having the property that there exists $n + 2$ points $\{\xi_k\}_{k=1}^{n+2}$ in E_a, such that $p_n(f, E_n) = p_n(f, \{\xi_k\}_{k=1}^{n+2})$. It is known [1, p. 22] that there always exists a set $D \subset E_a$, consisting of $n + k$ points, $2 \leqq k \leqq n + 3$, such that $p_n(f, E_a) = p_n(f, D)$. Furthermore, to this author's knowledge, every example of best uniform approximation to rational functions on infinite sets in the plane (e.g., [3], [4] and [5]) is one in which such a set consisting of $n + 2$ points exists or, can be shown equivalent to such an example.

2. Main theorem. Given $n + 2$ points $\{\xi_k\}_{k=1}^{n+2}$ in E_a let z_k be such that $\xi_k = z_k + a/z_k$, $|z_k| = 1$ and if $a = 1$, $0 \leqq \operatorname{Arg} z_k \leqq \pi$ for $k = 1, 2, \cdots, n + 2$. The z_k's are uniquely determined. Now let

$$\Phi_k = z_k^{-n/2} \prod_{\substack{j=1 \\ j \neq k}}^{n+2} [(z_k z_j - a)/|z_k z_j - a|] \text{ for}$$

$k = 1, 2, \cdots, n + 2$ where $0 \leqq \arg z^{1/2} < \pi$.