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THE TRANSLATIONAL HULL OF AN N-SEMIGROUP

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An N-semigroup is a commutative, cancellative, archimedean semigroup having no idempotents. In the first section of this paper the Tamura representation of an N-semigroup is used to determine the translational hull. The maximal semilattice decomposition of the translational hull is then investigated resulting in a complete determination of the classes of this decomposition in the case that the N-semigroup is power joined. These results are used in the second section which deals with ideal extensions of an N-semigroup by an abelian group, and ideal extensions of an abelian group by an N-semigroup. These extensions arise naturally in the maximal semilattice decomposition of a commutative separative semigroup. The latter part of this section contains results on cancellative extensions of N-semigroups, and a structure theorem of the class of weakly power joined, commutative, cancellative semigroups.

Notation and Preliminaries. Let S be a semigroup. We will write left [right] translations as operators on the left [right]; $\Lambda(S)$ $[P(S)]$ denotes the semigroup of all left [right] translations of S under the multiplication $(\lambda\lambda')x = \lambda(\lambda'x)$ $[x(pp') = (xp)p']$ for all $x \in S$. The translations $\lambda \in \Lambda(S)$ and $p \in P(S)$ are linked if $x(\lambda y) = (xp)y$ for all $x, y \in S$; $\Omega(S)$ denotes the translational hull of S, that is, the subsemigroup of $\Lambda(S) \times P(S)$ consisting of all pairs of linked translations. The subsemigroup of $\Omega(S)$ consisting of all pairs of the form (λ_a, p_a) is denoted by $\Pi(S)$. (Recall that $\lambda_a x = ax$ and $xp_a = xa$ for all $x \in S$).

We will need the following results concerning $\Omega(S)$ when S is a commutative cancellative semigroup. Proofs may be found in [6].

(1) $\Omega(S)$ is commutative and cancellative.

(2) $(\lambda, p) \in \Omega(S)$ if and only if $\lambda x = xp$ for all $x \in S$, and hence $\Omega(S) \cong \Lambda(S)$.

(3) $S \cong \Gamma(S)$ where $\Gamma(S) = \{\lambda_a \in \Lambda(S) \mid a \in S\}$.

We next describe the Tamura representation of an N-semigroup. Let N denote the positive integers and N_0 the nonnegative integers. Let G be an abelian group and $I\colon G \times G \to N_0$ be a function satisfying:

(i) $I(a, b) = I(b, a)$ $\qquad (a, b \in G)$,

(ii) $I(a, b) + I(ab, c) = I(a, bc) + (b, c)$ $\qquad (a, b, c \in G)$,

(iii) For each $a \in G$ there is an $m \in N$ such that $I(a^m, a) > 0$.

(iv) $I(e, e) = 1$ where e is the identity of G.