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ON THE DISTRIBUTIVITY OF THE LATTICE OF FILTERS OF A GROUPOID

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In this note we present the results announced in the Notices of the American Mathematical Society, January, 1969. Algebraic lattices are interesting and important algebraic structures. They occur in many branches of algebra, e.g. the lattice of all subalgebras of a universal algebra, the lattice of all filters of a groupoid, and the lattice of all ideals of a ring are all algebraic lattices. Moreover there is a natural connection between algebraic lattices and groupoids, since every algebraic lattice is isomorphic to the lattice of all filters of some groupoid, and in particular of the groupoid of all compact elements of the lattice. If an algebraic lattice is distributive, it is relatively pseudo-complemented and is a complete Brouwerian lattice in the sense of Garrett Birkhoff [1]. Hence it is natural to look for simple conditions on a groupoid that will insure that the lattice of its filters is distributive.

We show that the lattice of all filters is distributive if it is a sublattice of the lattice of all subgroupoids, but this condition is not always necessary for distributivity. If the groupoid is a semilattice, this condition is both necessary and sufficient. We then derive some conditions that are both necessary and sufficient for distributivity for groupoids. One of these is a modification of a condition given by Grätzer and Schmidt for semilattices.

A groupoid is a pair (G, τ) where G is a set and τ is a binary operation defined on G. We will use the following conventions:

(1) The operation τ will be called multiplication, and we will write ab for $a\tau b$.

(2) We will write simply G for (G, τ)

(3) The symbol 1 will denote the identity element of G if there is one.

(4) G^1 will denote the groupoid G with 1 adjoined if G has no identity element; otherwise $G = G^1$.

The operation in a groupoid G need not be associative or commutative, hence the elements $a_1, \cdots, a_n$ have in general many products which can be distinguished from each other by means of parentheses. We find it convenient to denote one of the products of elements $a_1, \cdots, a_n$ by $P(a_1, \cdots, a_n)$. We allow the elements to be used more than once and in any order. For example $P_1(a_1, a_2)$ might be $(a_1a_2)a_1$, while $P_2(a_1, a_2)$ could be $a_2((a_2a_1)a_1)$.