PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 3, 1972

MEROMORPHIC FUNCTIONS WITH NEGATIVE ZEROS AND POSITIVE POLES AND A THEOREM OF TEICHMULLER

JACK WILLIAMSON

Let $\mathscr{M}_\lambda$ denote the class of meromorphic functions of finite order λ whose zeros lie on the negative real axis and whose poles lie on the positive real axis. Let $\mathscr{T}_\lambda$ denote the class of functions belonging to $\mathscr{M}_\lambda$ whose zeros and poles are symmetrically located along the real axis.

In the study of certain aspects of the value distribution properties of meromorphic functions of order $\lambda < 1$, the class $\mathscr{M}_\lambda$, $\lambda < 1$, has recently been found to display certain striking and useful extremal properties, while earlier results on the subclass $\mathscr{T}_\lambda, \lambda < 1$, have been important as a guide to the possible values of their Nevanlinna deficiencies. In this note the class $\mathscr{T}_\lambda$, $\lambda > 1$, is studied and it is concluded that certain extremal properties displayed by $\mathscr{M}_\lambda$ for $\lambda < 1$ do not extend to the case $\lambda > 1$.

Introduction. This note is concerned with Nevanlinna's theory of meromorphic functions. We will assume familiarity with the standard notation and terminology of that theory. The order λ and the lower order μ of a meromorphic function f are defined by the familiar relations

$$\lambda = \lambda(f) = \limsup_{r\to\infty} \frac{\log T(r, f)}{\log r}; \ \mu = \mu(f) = \liminf_{r\to\infty} \frac{\log T(r, f)}{\log r}\,.$$

In 1939, Teichmüller [12] proved

THEOREM A. *Let $f \in \mathscr{M}_\lambda$ for $0 \leqq \lambda < 1$ and assume that the zeros $\{a_n\}$ and the poles $\{b_n\}$ of f satisfy*

$$(1) \qquad a_n = -\,b_n \qquad (n = 1, 2, \cdots)\,.$$

If

$$(2) \qquad u = 1 - \delta(0, f),\ v = 1 - \delta(\infty, f)$$

then

$$(3) \qquad u = v \geqq \cos\left(\frac{\pi\lambda}{2}\right)\,.$$

Although the hypothesis (1) of Teichmüller's theorem is quite restrictive, the theorem is important as a guide to possible relations