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THE p-CLASSES OF AN H^*-ALGEBRA

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This paper considers a family of $*$-subalgebras of a semisimple H^*-algebra A. For $0 < p \leqq \infty$ a nonnegative extended-real value $|a|_p$ is associated with each a in A; then the p-class A_p is defined to be $\{a \in A: |a|_p < \infty\}$. If $1 \leqq p \leqq \infty$, A_p is then a two-sided $*$-ideal of A (proper only if $p < 2$), and $(A_p, |\cdot|_p)$ is a normed $*$-algebra. $(A_2, |\cdot|_2)$ is $(A, ||\cdot||)$; and for $1 \leqq p < 2$, $(A_p, |\cdot|_p)$ is a Banach $*$-algebra, for which structure theorems are given.

1. Introduction. Let A be a semisimple H^*-algebra with inner product and norm denoted by $(\ ,\ )$ and $||\cdot||$, respectively. The trace class of A, that is, the set $\tau(A) = \{xy: x, y \in A\}$, has been studied by Saworotnow and Friedell [8], who show, first of all, that for any nonzero $a \in A$ there exists a positive element $[a] \in A$ such that $[a]^2 = a^*a$, and $a \in \tau(A)$ if and only if $[a] \in \tau(A)$. An algebra norm τ is then introduced on $\tau(A)$ by defining $\tau(a) = tr[a]$ for each $a \in \tau(A)$, where in turn the trace functional tr is unambiguously defined on $\tau(A)$ by letting $tr\, xy = (x, y^*) = \Sigma(xyp_\omega, p_\omega)$, $\{p_\omega: \omega \in \Omega\}$ being any maximal family of mutually orthogonal nonzero self-adjoint idempotents. With this norm, $\tau(A)$ is actually a Banach algebra [9, Corollary to Theorem 1]. This presentation parallels that of Schatten [10] for τc, the trace class of σc, the Schmidt class of operators on a Hilbert space.

In a somewhat similar sense our central development in §3 brings over into the present context some of the work of McCarthy [6] on the operator algebras c_p. We preface this with a basic spectral theorem established in §2; in §4 we study the structure of the Banach $*$-algebras A_p, where $1 \leqq p < 2$. Finally, in §5 we relate A_p to the class c_p of operators on a Hilbert space [6; 2, ch. XI. 9] and also to $\mathscr{C}_p$ spaces [3, pp. 70 ff.; 5].

2. Preliminary spectral theory. Throughout the remainder of this paper A will continue to denote a semisimple H^*-algebra. By a *projection* p in A we shall mean a nonzero self-adjoint idempotent. A projection p is *primitive* if p cannot be expressed as $p = p_1 + p_2$, where p_1 and p_2 are orthogonal projections. By a *projection base* in A we mean a maximal family of mutually orthogonal projections (not necessarily primitive); note that if $a \in A$ and $\{p_\omega: \omega \in \Omega\}$ is a projection base, then $a = \Sigma a p_\omega = \Sigma p_\omega a$ [1, Theorem 4.1, where primitivity of the projections is not needed to establish this point]. Finally, we shall say that an element a in A is *positive* if $(ax, x) \geqq 0$ for every $x \in A$; a is then necessarily self-adjoint.