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CANONICAL FORMS FOR LOCAL DERIVATIONS

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Consider a field k, the formal power series field $k((x))$ in one variable over k, and a derivation D of $k((x))$ that maps k into itself. We wish to replace x by another generator y of $k((x))$ so that Dy has a particularly simple expression as a function of y. This is accomplished subject to certain restrictions on the differential field k, some deductions are drawn, and there are extensions to the analogous problem for power series rings in several variables.

We first consider a derivation D on a noetherian local ring R. If M is the maximal ideal of R, then for any $N = 1, 2, \cdots$ we have $DM^N \subset M^{N-1}$. Thus D is automatically continuous in the natural topological ring structure of R, where a basis for the neighborhoods of zero are the various powers of M.

THEOREM 1. *Let R be a complete noetherian local ring containing $\boldsymbol{Q}$, M the maximal ideal of R and D a derivation of R such that $DM \not\subset M$. Then M has a set of generators $y_1, \cdots, y_n$ such that $Dy_1 = \cdots = Dy_n = 1$.*

There is an element $x \in M$ such that $Dx \notin M$. For any other element $y \in M$, either Dy or $D(x + y)$ is not in M. Since x and y generate the same ideal in R as do x and $x + y$, it follows that M is generated by those of its elements x for which $Dx \notin M$, that is, for which Dx is a unit in R. Now if $x \in M$ and $Dx \notin M$, we have $D(x/Dx) - 1 = xD(1/Dx) \in M$, so that M is generated by those of its elements x satisfying $Dx - 1 \in M$. Since R is noetherian, a finite number of such x's, say $x_1, \cdots, x_n$, will generate M. If we have elements $y_1, \cdots, y_n \in M$ such that $x_i - y_i \in M^2$ for each $i = 1, \cdots, n$ then $y_1, \cdots, y_n$ also generate M, and we shall be done with the proof if we can find such $y_1, \cdots, y_n$ such that $Dy_1 = \cdots = Dy_n = 1$. To do this, we shall show by a successive approximation process that $x_1, \cdots, x_n$ may be replaced by elements which differ from these by elements in successively higher powers of M in such a way that the new $Dx_1 - 1, \cdots, Dx_n - 1$ also belong to high powers of M, and we shall then let each y_i, $i = 1, \cdots, n$, be the limit of the sequence of x_i's thus obtained. Specifically, we are reduced to showing that if $x_1, \cdots, x_n$ generate M and $N \geqq 1$ is an integer such that for each $i = 1, \cdots, n$ we have $Dx_i - 1 \in M^N$, then there exist $z_1, \cdots, z_n \in M^{N+1}$ such that each $D(x_i + z_i) - 1 \in M^{N+1}$. Since $DM^{N+1} \subset M^N$, it suffices