PACIFIC JOURNAL OF MATHEMATICS
Vol. 42, No. 3, 1972

A PROBABILISTIC METHOD FOR THE RATE OF CONVERGENCE TO THE DIRICHLET PROBLEM

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The expectation $E^P(\Phi)$ approximates the solution $u(z) = E^W(\Phi)$ of the Dirichlet problem for a plane domain D with boundary conditions ϕ on the boundary γ of D, where W is Wiener measure, P is the measure generated by a random walk which approximates Brownian motion beginning at z, and Φ is the functional on paths which equals the value of ϕ at the point where the path first meets γ. This paper develops a specific rate of convergence. If γ is C^2, and P^n is generated by random walks beginning at z, with independent increments in the coordinate directions at intervals $1/n$, with mean zero, variance $1/\sqrt{n}$, and absolute third moment bounded by M, then $|E^{P^n}(\Phi) - E^W(\Phi)| \leqq (CMV/\rho(z, \gamma))n^{-1/16}(\log n)^{9/8}$, where V is the total variation of ϕ on γ, $\rho(z, \gamma)$ is the distance from z to γ, and C is a constant depending only on γ.

Assume D is a Jordan region. If $z_t = x_t + iy_t$ is Brownian motion in R^2 beginning at z_0, (cf. e.g., [5, p. 262]), and $\tau = \inf\{t: z_t \in \gamma\}$ is the first time z hits the boundary γ of D, then Φ is the functional given by $\Phi(z.) = \phi(z_\tau)$. Let $E^W(\Phi(z.)) = \int \Phi(z.)dW$ be the expectation of Φ with respect to Wiener measure W on $C([0, \infty), \mathbb{C})$. (See [8, pp. 218–19] for a definition of Brownian motion on the interval [0, 1] and the corresponding Wiener measure.)

Let $g_1^1, g_1^2, g_2^1, g_2^2, \cdots, g_k^1, g_k^2, \cdots$ be a sequence of indendent random variables with mean zero, variance 1, and absolute third moment bounded uniformly by $M < \infty$, and let

$$\xi_i^\alpha = g_i^\alpha/\sqrt{n},\ \zeta_0 = z_0,\ \zeta_k = z_0 + \sum_{i=1}^k (\xi_i^1 + \sqrt{-1}\xi_i^2),\ t_k = k/n\ .$$

Let $\xi(t)$ be the continuous random broken line which has vertices (t_k, ζ_k) and is linear between vertices. Let P^n be the measure on $C([0, \infty), \mathbb{C})$ generated by this line, i.e., $P^n(S) = P(\xi(t) \in S)$.

Now by the Central Limit Theorem $P^n(\xi^\alpha(t) \leqq \lambda) \rightarrow W(z_t^\alpha \leqq \lambda)$, $\alpha = 1, 2$, where $\xi^\alpha(t)$, z_t^α are the real and imaginary parts of $\xi(t)$, z_t respectively, (cf. e.g., [1, pp. 186–7]). More exactly one has the Barry-Esseen Theorem [3, p. 521]: *For nt an integer*

$$\sup_\lambda |P(\xi^\alpha(t) \leqq \lambda) - N(\lambda/\sqrt{t})| \leqq \frac{33}{4}M/\sqrt{nt} \tag{1.1}$$

where $N(x)$ is the normal distribution. A useful generalization of the