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ON BOREL PRODUCT MEASURES

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It has been known for many years that the product of two regular borel measures on compact hausdorff topological spaces may not be borel in the product topology. The problem of defining a new product measure that extends the classical product measure and carries over this borel property has been approached in different ways by Edwards, by Bledsoe and Morse (Product Measures, Trans. Amer. Math. Soc. 79 (1955), 173–215; called PM here.) and by Johnson and Berberian. Godfrey and Sion and Hall have shown that all three of these methods are equivalent for the case of Radon measures on locally compact hausdorff spaces.

Elliott has extended the results of PM by defining a product measure for a pair, the first of which is a (generalized) borel measure and the second a continuous regular conditional measure (generalization of conditional probability), and proving a corresponding Fubini-type theorem.

The purpose of this paper is to extend the results of PM in a manner similar to Elliott's, but with his continuity condition replaced by an absolute continuity condition and by a "separation of variables" condition. It is still an open question whether Elliott's continuity condition is necessary.

1. DEFINITIONS AND NOTATION[1]. By a *measure* (outer measure) μ on a space M is meant a nonnegative countably subadditive function on 2^M, the subsets of M. In a topological space (M, m), an *m-borel measure on M* is any measure on M for which the open sets are (Caratheodory) measurable, and the *borel sets* of (M, m) are the members of the smallest σ-algebra containing m. If G is any family of sets, let σG be the union $\bigcup_{a \in G} a$ of the family G. If $H \subseteqq 2^M$ and g is a nonnegative function on H, then $mss\, gMH$ is defined to be the function on 2^M such that $mss\, gMH(A) = \inf_G \sum_{a \in G} g(a)$ where G varies over all countable subsets of H for which $A \subseteqq \sigma G$. $\psi = mss\, gMH$ is called the *measure generated by the gauge g*, and H is called the *basis of* ψ.

2. Product measures. If μ measures M and ν measures N, we call a subset D of $M \times N$ a *nilset* (more correctly a $\mu\nu$-*nilset*) if

$$\iint Cr_D(x, y)\mu dx \nu dy = 0 = \iint Cr_D(x, y)\nu dy \mu dx ,$$

where Cr_D is the characteristic function on D. The *ordinary product measure* of μ *and* ν *is given by*

[1] Most of the notation used here is taken from [2] and [4].