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NOTES ON RELATED STRUCTURES OF A UNIVERSAL ALGEBRA

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The related structures of a universal algebra $\mathfrak{A}$ that are studied here are the subalgebra lattice of $\mathfrak{A}$, the congruence lattice of $\mathfrak{A}$, the automorphism group of $\mathfrak{A}$, and the endomorphism semigroup of $\mathfrak{A}$. Characterizations of these structures known, and E. T. Schmidt proved the independence of the automorphism group and the subalgebra lattice. It has been conjectured that the first three of the structures listed above are independent, i.e., that the congruence lattice, subalgebra lattice, and automorphism group are independent. One result in this paper is a proof of a special case of this conjecture. Various observations concerning the relationship between the endomorphism semigroup and the congruence lattice are also in this paper. In the last section a problem of G. Grätzer is solved, namely that of characterizing the endomorphism semigroups of simple unary algebras. (An algebra is simple when the only congruences are the trivial ones.)

The characterizations of the various related structures are as follows: the congruence lattice is an arbitrary algebraic lattice the subalgebra lattice is an arbitrary algebraic lattice; the automorphism group is an arbitrary group; the endomorphism semigroup is an arbitrary semigroup with identity. The "independence of the automorphism group and the subalgebra lattice" is more precisely phrased as: for each pair $\langle \mathfrak{G}, \mathfrak{L} \rangle$, where $\mathfrak{G}$ is a group and $\mathfrak{L}$ is an algebraic lattice with more than one element, there is an algebra $\mathfrak{A}$ with $\mathfrak{G}$ isomorphic to the automorphism group of $\mathfrak{A}$ and with $\mathfrak{L}$ isomorphic to the subalgebra lattice of the same algebra $\mathfrak{A}$. All statements about the independence of related structures will be phrased in this way.

Mentioned above was a proof of a special case of the independence of the triple consisting of the automorphism group, the subalgebra lattice, and the congruence lattice. As a corollary one gets a proof of a special case of the independence of the pair consisting of the automorphism group and the congruence lattice. E. T. Schmidt published what was supposed to be a proof of the independence of this pair of structures. But, his proof [10] was incorrect. (See e.g. Exercise 31 of chapter 2 of [2]). The author has just completed a proof of the independence of this pair [8].

The terminology essentially conforms to that in [2]. ω(or ω_A) will denote the equality relation on the set A, and ι(or ι_A) will denote