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# SUPERADDITIVITY INTERVALS AND BOAS' TEST

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**A test is given for determining maximal intervals of superadditivity for convexo-concave functions. The test is then applied to several families of ogive-shaped functions.**

1. Superadditive functions have been widely studied [8, 11] for their own sake but have also found important applications in reliability theory, e.g. [6]. However, tests for superadditivity were non existent in the literature until Bruckner's work [3] in 1962. A more constructive (hence more readily applicable) test due to Boas was given in 1964 in a paper by Beckenbach [2] on analytic inequalities, an area where superadditivity is of use (see [2] for a derivation of Whittaker's inequality [12]). Boas' test is here viewed in the light of Bruckner's result, strengthened, and applied to some families of convexo-concave functions as suggested in [2].

2. Consider a continuous, real-valued function, $f$, of a real variable, $x \in \boldsymbol{R}$. Then $f$ is called "superadditive" on $[\beta, b] \subset \boldsymbol{R}$ if

$$f(x) + f(y) \leqq f(x + y)$$

for every $x, y, x + y$ in $[\beta, b]$. We normalize to the cases $\beta = 0$, $b > 0$. In this event, superadditivity implies $f(0) \leqq 0$. The following sufficient condition for superadditivity is due to Boas [2]:

THEOREM (Boas' Test). *Assume $f$ is nonnegative on $[0, b]$ with $f(0) = 0$ and $f$ has a continuous derivative on $[0, b]$. If there are numbers $a \leqq b/2$ and $c \leqq a$ such that*

(0) *$f$ is star-shaped[1] on $[0, 2a]$,*

(i) *$f$ is concave[2] and satisfies $f(x/2) \leqq f(x)/2$ on $[c, b]$,*

(ii) *$f'(0) < f'(b)$,*

(iii) *$f'(x) - f'(b - x)$ has at most one zero in $(0, a)$.*

*Then $f$ is superadditive on $[0, b]$.*

A proof of the theorem can be made by considering separately the cases:

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[1] $f$ is "star-shaped" on $[0, A]$ means for every $x \in [0, A]$, and every $\alpha \in [0, 1]$ it is true that $f(\alpha x) \leqq \alpha f(x)$. For $f \in C^1[0, A]$ it is necessary and sufficient [4] that $f'(x) \geqq f(x)/x$ for all $x \in (0, A]$.

[2] The function $f$ is called "convex" on $[a, b]$ if for every $x, y \in [a, b]$ it is true that $f((x+y)/2) \leqq (f(x)+f(y))/2$; $f$ is called "concave" if $-f$ is convex.