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CHARACTERIZATIONS OF AMENABLE AND STRONGLY AMENABLE C^*-ALGEBRAS

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In this paper it is proved that a C^*-algebra A is strongly amenable iff A satisfies a certain fixed point property when acting on a compact convex set, or iff a certain Hahn-Banach type extension theorem is true for all Banach A-modules. It is proved that a C^*-algebra A is amenable iff A satisfies a weaker Hahn-Banach type extension theorem.

A topological group G is said to be amenable if there is a left invariant mean on the space of bounded continuous complex functions on G. A number of papers have been published which give equivalent definitions of amenability (for example, see the papers [4, 7, 11] or the book [3]). It has recently been proven that a locally compact group G is amenable iff for all two-sided $L^1(G)$-modules X and bounded derivations D of $L^1(G)$ into X^*, we have that D is the inner derivation induced by an element of X^* [5, Theorem 2.5]. This result motivates the definition of amenable and strongly amenable C^*-algebras [5, sections 5 and 7]. In §2 of this paper we give some conditions on a C^*-algebra that are equivalent to amenability or strong amenability and are analogous to some of the known equivalent definitions of amenable groups. In §3 we show that the generalized Stone-Weierstrass theorem for separable C^*-algebras is true when the C^*-subalgebra in question is strongly amenable.

1. Preliminaries. Let A be a C^*-algebra. Then a complex Banach space X is called a Banach A-module if it is a two-sided A-module and there exists a positive real number M such that for all $a \in A$ and $x \in X$ we have

$$||ax|| \leqq M||a||\,||x||$$

and

$$||xa|| \leqq M||x||\,||a|| .$$

If X is a Banach A-module, then the dual space X^* becomes a Banach A-module if we define for $a \in A,, f \in X^*$, and $x \in X$,

$$(af)(x) = f(xa)$$
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A derivation from A into X^* is a bounded linear map D from A into