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VISIBILITY MANIFOLDS

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Several of the basic features of automorphic function theory—notably the notion of limit set—can be extended to apply to the study of Riemannian manifolds M of nonpositive curvature. Under somewhat stronger curvature conditions (e.g. $K \leqq c < 0$) M is called a *Visibility manifold*. For such manifolds there results a classification into three types: *parabolic*, *axial*, and *fuchsian*. This trichotomy is closely related to many of the most basic topological and geometric properties of M, and such relationships will be examined in some detail. For example, the trichotomy may be expressed in terms of the number (suitably counted) of closed geodesics in M, namely: 0, 1, or ∞. As to methodology: the conventional analytic machinery of C^∞ Riemannian geometry is used, at least initially; however, at many crucial points it will be the qualitative behavior of geodesics (âla Busemann) that is important.

In the late nineteenth century Poincaré discovered an important link between geometry and automorphic function theory, namely that the open disk P in the complex plane admits a Riemannian metric such that (1) P has constant negative curvature and gives a model for the noneuclidean geometry of Bolyai and Lobachevsky; and (2) the orientation preserving isometries of P are exactly the linear fractional transformations that preserve P.

Automorphic function theory makes essential use of an extrinsic feature of the disk: its boundary circle S^1 in the plane. For example, a linear fractional transformation φ as in (2) that has no fixed points in P must have either one or two fixed points in S^1. Also, if D is a properly discontinuous group of such transformations then the accumulation points of any orbit $D(p)$, $p \in P$, form a closed D-invariant subset $L(D)$ of S^1 called the *limit set* of D. Analysis of the limit set $L(D)$ and of the fixed points of the elements of D gives extensive information about the Riemann surface $M = P/D$. When the Poincaré metric was introduced on P this approach also yielded properties of the geodesic flow on surfaces M of constant negative curvature. But since the method was tied to complex analysis it was not clear how to extend it to manifolds of higher dimension and variable curvature.

Our object in this paper is to study Riemannian manifolds of sectional curvature $K \leqq 0$ by generalizing, or rather geometrizing, some of these basic features of automorphic function theory. Thus