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ON k-SPACES, k_R-SPACES AND $k(X)$

E. MICHAEL

Two examples of k_R-spaces which are not k-spaces are constructed; one of them is a σ-compact cosmic space, and the other is an $\aleph_0$-space. On the positive side, a theorem is proved which implies that every σ-compact $\aleph_0$-space which is a k_R-space must be a k-space.

1. Introduction. In this paper, we prove and extend some results which were announced in [8].

Recall that a topological space X is called a *k-space* if every subset of X, whose intersection with every compact $K \subset X$ is relatively open in K, is open in X. (For example, locally compact spaces and first-countable spaces are k-spaces.) Analogously, a space X is a *k_R-space* if it is completely regular and if every $f: X \to R$, whose restriction to every compact $K \subset X$ is continuous, is continuous on X. Clearly every completely regular k-space is a k_R-space. The converse is false, as was first shown by an example of M. Katětov which appeared in a paper by V. Pták [15, p. 357]. That example, however, was not normal[1], and our first purpose in this note is to construct two examples which *are* normal,—in fact, regular Lindelöf and thus paracompact. Both our examples are modifications of Katětov's example, which had, in turn, been previously introduced (for a different purpose) by J. Novák in [12].

Before giving more details, let us review some definitions. A covering $\mathscr{A}$ of a space X is a *network* (resp. *pseudobase*) for X if, wherever $C \subset U$ with C a singleton (resp. compact) and U open in X, then $C \subset A \subset U$ for some $A \in \mathscr{A}$. A regular space with a countable network (resp. pseudobase) is called *cosmic* (resp. an *$\aleph_0$-space*). It is shown in [7, Proposition 10.2 and Corollary 11.5] that a regular space is a continuous (resp. quotient) image of a separable metric space if and only if it is cosmic (resp. an $\aleph_0$-space and a k-space). We clearly have

$$\text{separable metric} \to \aleph_0 \to \text{cosmic} \to \text{regular Lindelöf},$$

and none of these implications is reversible.

We can now describe the principal features of our examples, as follows.

EXAMPLE 1.1. There exists a σ-compact, cosmic k_R-space which

[1] It has a countable dense subset and a closed, discrete subset of cardinality c, and is thus not normal by a result of F. B. Jones [4].