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PRODUCT INTEGRALS FOR AN ORDINARY DIFFERENTIAL EQUATION IN A BANACH SPACE

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Let Y be a Banach space with norm $|\ |$, and let R^+ be the interval $[0, \infty)$. Let A be a function on R^+ having the properties that if t is in R^+ then $A(t)$ is a function from Y to Y and that the function from $R^+ \times Y$ to Y described by $(t, x) \to A(t)[x]$ is continuous. Suppose there is a continuous real-valued function α on R^+ such that if t is in R^+ then $A(t) - \alpha(t)I$ is dissipative. Now it is known that if z is in Y, the differential equation $u'(t) = A(t)[u(t)]$; $u(0) = z$ has exactly one solution on R^+. It is shown in this paper that if t is in R^+ then $u(t) = {}_0\prod^t \exp[(ds)A(s)][z] = {}_0\prod^t[I - (ds)A(s)]^{-1}[z]$, where the exponentials are defined by the solutions of the associated family of autonomous equations.

The dissipitavity condition on A is simply that if (t, x, y) is in $R^+ \times Y \times Y$ and c is a positive number then

$$|[I - cA(t)][x] - [I - cA(t)][y]| \geqq [1 - c\alpha(t)]|x - y| . \tag{1}$$

The author and R. H. Martin, Jr. [5] have shown that if (1) holds, and z is in Y, then there is exactly one continuously differentiable function u from R^+ to Y such that

$$u(0) = z \tag{2}$$

and

$$u'(t) = A(t)[u(t)] \tag{3}$$

whenever t is in $(0, \infty)$. In the present article we shall show that u can be expressed as a product integral in each of two forms:

$$u(t) = \prod_0^t \exp[(ds)A(s)][z] \tag{4}$$

and

$$u(t) = \prod_0^t [I - (ds)A(s)]^{-1}[z] . \tag{5}$$

Our work is related to results of J. V. Herod [2, §6] and G. F. Webb [7], [8]. Herod showed that representation (5) is valid if the mapping $(t, x) \to A(t)[x]$ is bounded on bounded subsets of $R^+ \times Y$. Webb obtained in [7] a representation similar to (4) under a set of hypotheses different from, and independent of, those used here. In