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ON 2-TRANSITIVE COLLINEATION GROUPS OF FINITE PROJECTIVE SPACES

WILLIAM M. KANTOR

In 1961, A. Wagner proposed the problem of determining all the subgroups of $P\Gamma L(n, q)$ which are 2-transitive on the points of the projective space $PG(n-1, q)$, where $n \geqq 3$. The only known groups with this property are: those containing $PSL(n, q)$, and subgroups of $PSL(4, 2)$ isomorphic to A_7. It seems unlikely that there are others. Wagner proved that this is the case when $n \leqq 5$. In unpublished work, D. G. Higman handled the cases $n = 6, 7$. We will inch up to $n \leqq 9$. Our result is that nothing surprising happens. The same is true if $n = r^\alpha + 1$ for a prime divisor r of $q - 1$.

One of Wagner's results is that it suffices to only consider subgroups of $PGL(n, q)$. Once this is done, it becomes simpler to view the problem as one concerning linear groups: find all those subgroups G of $GL(n, q)$ which are 2-transitive on the 1-spaces of the underlying vector space V. Our approach is based primarily on three facts. (1) Wagner showed that the global stabilizer in G of any 3-space of V induces at least $SL(3, q)$ on that 3-space. (2) Unless $G \geqq SL(n, q)$ or $n = 4$, $q = 2$, and $G \approx A_7$, no nontrivial element of G can fix every 1-space of some n-2-space of V. (3) $G \leqq SL(n, q)$ if $|G|$ is divisible by a prime which is a primitive divisor of $q^m - 1$ for a suitable $m \leqq n - 2$.

Wagner's results are in [10]. Higman's result, and the case $n = 2^\alpha + 1$ and q odd, are mentioned by Dembowski [1], p. 39. The result mentioned above in (2) is an easy consequence of results of Wagner. The idea used in (3) is due to Perin [8] and, independently, to G. Hare and E. Shult.

I am indebted to G. Seitz for several helpful remarks.

2. Notation and preliminaries. As already mentioned, we will be dealing with linear groups. Let V be an n-dimensional vector space over $GF(q)$. We write $GL(V) = GL(n, q)$ and $SL(V) = SL(n, q)$. It will be convenient to regard everything as taking place in the relative holomorphic $V \cdot GL(V)$. For any subgroups K, L of this semidirect product we can then consider the normalizer $N_L(K)$ and centralizer $C_L(K)$. If $L \leqq GL(V)$ and W is an L-invariant subspace of V, we write $L^W = L/C_L(W)$ for the subgroup of $GL(W)$ induced by L. $C_L(V/W)$ and $L^{V/W}$ are defined similarly. For any group G, as usual G' is its commutator subgroup, $Z(G)$ its center, and $\Phi(G)$ its Frattini subgroup.